Drawing an analogy between the Dirac-Maxwell-Einstein theory and a field model for semions

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It is shown that there is much in common between the Dirac-Maxwell-Einstein (DME) theory and a semion model in two-dimensional Grassmann space which allows one to treat a supersymmetric semion model in $D=2+1$ space-time[1] as an effective theory resulting from the two-dimensional model.

Semions (i.e., particles with spin 1/4 and 3/4 in $D=2+1$), which are special representatives of the anyons,[2] reveal interesting field-theoretical[3,4] and interaction[5] properties and, one hopes, are relevant to high-T_c superconductivity and to strongly correlated quantum electron systems in two-dimensional space.[6]

In the present letter we construct, by drawing an analogy with the DME theory, a field-theoretical semion model in two-dimensional Grassmann space which is effectively equivalent to a $D=2+1$ supersymmetric semion model[1] in the momentum representation, with space-time dynamical characteristics of the latter arising as a manifestation of a gauge and a gravitational field of the 2D model.

To clarify this idea, we consider the grounds of both the DME theory and the semion model in parallel. The following notation and convention are used: $(l,m,n...)$ stand for vector indices, in particular, in $D=2+1$ space-time; $(\alpha,\beta,\gamma,...=1,2)$ are $SL(2,R)$ spinor indices, which are raised and lowered by the unit skew-symmetric matrices $\varepsilon^{\alpha\beta}=\varepsilon_{\alpha\beta}$; and the signature of the Minkowski space-time metric g_{mn} is chosen to be $(+,-...,-)$.

Lorentz group representation. To describe fermions, we introduce the D-dimensional vector matrices γ^m which generate the Clifford algebra

$$\{\gamma^m,\gamma^n\}\equiv\gamma^m\gamma^n+\gamma^n\gamma^m=2g^{mn}. \tag{1}$$

The generators of the spinor representation of the Lorentz group are realized as the commutators of γ^m:

$$S^{mn}=\frac{i}{4}[\gamma^m,\gamma^n]. \tag{2}$$

To describe semions, we introduce the spinor operators L^α which generate the Heisenberg algebra[4,7,8]

$$[L^\alpha, L^\beta]\equiv L^\alpha L^\beta-L^\beta L^\alpha=i\varepsilon^{\alpha\beta}. \tag{1a}$$

The generators of $SL(2,R)$ (which are isomorphic to the $D=2+1$ Lorentz group) are realized for the case of semions as the anti-commutators of L^α:

$$S^{\alpha\beta}=\frac{1}{2}\{L^{\alpha}, L^{\beta}\}. \quad (2a)$$

An irreducible representation of the Heisenberg-Weyl group generated by the algebra (1a) splits into irreducible $SL(2,R)$ representations with the weights 1/4 and 3/4 describing semions (see Refs. 1 and 4 for details).

In view of the anti-commutator in (2a), we consider L^{α} to be odd and the relative statistics of the $SL(2,R)$ representations to be fermionic.

One can see that γ^{m} and L^{α} are the antipodes in the sense that where the commutator arises for the Dirac matrices the anti-commutator arises for L^{α} and vice versa. Below we use this interchange in the commuting and anti-commuting properties for constructing the semion model by analogy with the DME theory.

Free field equations. Fermions are known to propagate in the space-time, which is parametrized by bosonic vector coordinates x^{m}, and to be described by the Dirac equation

$$\gamma^{m}\partial_{m}\psi(x)=0, \quad (3)$$

where $\psi_{\alpha}(x)$ is a fermion wave function. Here we restrict the consideration to the massless fermions. See below a comment on the analogy between the massive Dirac theory and the semion model.

Semions are assumed to propagate in the Grassmann spinor space parametrized by odd Majorana coordinates θ^{α} and described by the equation

$$L^{\alpha}\frac{\partial}{\partial\theta^{\alpha}}\Phi(\theta)=0, \quad (3a)$$

where $\Phi(\theta)=A+i\theta^{\alpha}\psi_{\alpha}+i\theta^{\alpha}\theta_{\alpha}C$ is a semion wave function which transforms as the Heisenberg–Weyl group representation.

Interaction with an Abelian gauge field. To consider the interaction of fermions with an (external) Maxwell field $A_{m}(x)$, we write the Dirac equation in the form

$$\gamma^{m}D_{m}\psi\equiv\gamma^{m}[\partial_{m}+iA_{m}(x)]\psi(x)=0, \quad (4)$$

which is covariant under the Abelian gauge transformations

$$\psi(x)\rightarrow\psi(x)\exp i\varphi(x),$$

$$A_{m}(x)\rightarrow A_{m}(x)-\partial_{m}\varphi(x). \quad (5)$$

In the same way, we introduce the interaction of semions with an odd Abelian gauge field $A_{\alpha}(\theta)=a_{\alpha}+\theta^{\beta}p_{\beta\alpha}+i\theta^{\beta}\theta_{\beta}c_{\alpha}$ by generalizing Eq. (3a) to the form

$$L^{\alpha}D_{\alpha}\Phi(\theta)\equiv L^{\alpha}[\partial_{\alpha}+A_{\alpha}(\theta)]\Phi(\theta)=0, \quad (4a)$$

which is covariant under the Abelian gauge transformations

$$\Phi(\theta)\rightarrow\Phi(\theta)\exp i\varphi(\theta),$$

$$A_{\alpha}(\theta)\rightarrow A_{\alpha}(\theta)-i\partial_{\alpha}\varphi(\theta). \quad (5a)$$

Note that in an appropriately chosen gauge $A_\alpha(\theta)$ takes the form

$$A_\alpha(\theta)=\theta^\beta p_{\beta\alpha}+i\theta^\beta\theta_\beta c_\alpha,$$

where $p_{\alpha\beta}=p_{\beta\alpha}=i\gamma^m_{\alpha\beta}p_m$, and p_m is an even vector.

If $A_m(x)$ and $A_\alpha(\theta)$ are considered to be free fields, then they satisfy this condition.

Free Maxwell's equations.

$$\partial_m F^{mn}(x)=0, \tag{6}$$

where $F^{mn}(x)=\partial_m A_n(x)-\partial_n A_m(x)$.

$$\partial_\alpha F^{\alpha\beta}(\theta)=0, \tag{6a}$$

where $F^{\alpha\beta}(\theta)=\partial_\alpha A_\beta(\theta)+\partial_\beta A_\alpha(\theta)=p_{\alpha\beta}+p_{\beta\alpha}+2i(\theta_\alpha c_\beta+\theta_\beta c_\alpha)$. Therefore, $A_\alpha(\theta)$ with $c_\alpha=0$ is the solution of Eq. (6a). Substituting this solution into Eq. (4a) and choosing the gauge mentioned above, we obtain the equation

$$L^\alpha D_\alpha\Phi(\theta)=0, \tag{4b}$$

where $D_\alpha=\partial/\partial\theta^\alpha+\theta^\beta p_{\beta\alpha}$. One may recognize the supercovariant derivative of the $N=1$, $D=2+1$ SUSY theory[9] in the momentum representation, where the role of the momentum is played by the corresponding A_α component.

The integrability conditions for Eqs. (4) and (4b) are

$$D_m D^m\psi(x)+\frac{i}{2}F_{mn}\gamma^m\gamma^n\psi(x)=0. \tag{7}$$

The second term in (7) describes the electromagnetic interaction caused by the nonzero fermion magnetic moment,

$$D_\alpha D^\alpha\Phi(\theta)-2ip_{\alpha\beta}L^\alpha L^\beta\Phi(\theta)=0. \tag{7a}$$

An analogous term is present in Eq. (7a), but since now we treat $p_{\alpha\beta}$ as the $D=2+1$ semion momentum and in view of Eq. (2a), $p_{\alpha\beta}L^\alpha L^\beta$ is interpreted as the Pauli-Lubanski Casimir operator of the $D=2+1$ (super) Poincaré group. If $\Phi(\theta)$ is an eigenfunction of this operator, Eq. (7a) splits into

$$p_{\alpha\beta}\,L^\alpha L^\beta\Phi(\theta)=m\Phi(\theta)$$

and

$$(D_\alpha D^\alpha-2im)\Phi(\theta)=0, \tag{7b}$$

which describe, in the momentum representation, the propagation of a free $N=1$, $D=2+1$ SUSY semion with a mass m and the lowest superhelicity 1/4 ($p_{\alpha\beta}$ in this case satisfies the mass-shell condition[1] $p_{\alpha\beta}p^{\alpha\beta}=2m^2$).

Note that if we take into account the "massive" term in Eq. (4b), i.e., draw an analogy with the massive Dirac equation $(i\gamma^m\partial_m-m)\psi(x)=0$, then we would obtain a semion superfield which possesses an arbitrary superhelicity $\frac{1}{2}(\frac{1}{2}+n)$ $(n=0,1\ldots\infty)$, where the "mass" parameter is proportional to the superhelicity.

Without going into further details, we assert that to construct a semion action from which Eqs. (7b) arise we should take into account the gravity in the Grassmann space,[10] where the analogy with the DME theory reaches its "top."

Dirac-Maxwell-Einstein action.

$$S=\int d^3x\sqrt{g}\left[\psi\gamma^{(m)}e^{n}_{(m)}\hat{D}_n\psi-\left(\frac{1}{4}F_{mn}F^{mn}+R(x)+\lambda\right)\right], \tag{8}$$

where $g_{mn}(x)$ is a Riemann metric, $\sqrt{g}\equiv\det(g_{mn}(x))$, $e^{n}_{(m)}$ is the vielbein [(m) corresponds to the local tagent space], $\hat{D}_n$ contains the spin connection, $R(x)$ is the Riemann scalar curvature, and λ is a cosmological constant.

Semion action.

$$S=\int d^3\theta G(\theta)\left[i\Phi^{\dagger}L^{(\alpha)}e^{\beta}_{(\alpha)}\hat{D}_{\beta}\Phi-\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+R(\theta)-6m^2\right)\right], \tag{8a}$$

where $g_{\alpha\beta}(\theta)=[1/G(\theta)]\varepsilon\alpha\beta$ is a Riemann metric of the Grassmann space, $e^{\beta}_{(\alpha)}$ is the corresponding vielbein [(α) corresponds to the local tangent space], $\hat{D}_{\alpha}$ contains the spin connection, $R(\theta)=3i\partial_{\alpha}\partial^{\alpha}G(\theta)$ is the Riemann scalar curvature, and $6m^2$ is a cosmological constant (which determines the semion mass) whose sign is fixed by the requirement for $p_{\alpha\beta}$ to be time-like (the absence of tachions). Note that the action is even since L^{α} changes the statistics of the function it is acting on (see above), i.e., it effectively anticommutes with D_{α}. Here $\Phi(\theta)$ is the representation of the supergroup, which is realized on L^{α}, rather than the representation of the Heisenberg–Weyl group.

We assume that Eq. (8a) may be considered as the action of the effective $D=2+1$, $N=1$ SUSY semion model in the momentum representation, and that the coordinate representation can be obtained by a functional integration of (8a) over all independent configurations of the gauge field.

The semion model which we proposed is essentially based on the analogy with, and uses the principles of, the construction of the DME theory, which allows one to provide all fields of the model with clear gauge and geometrical meaning. This model seems to be useful for the development of the field-theoretical approach to anyons,[1,4,11] which, in particular, encounters problems in constructing equations of motion and actions due to the absence of a reliable geometric and symmetry basis.

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