

Local gauge and UV divergences

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In an invariant regularization of the Yang–Mills Lagrangian which eliminates UV divergences, the Gaussian perturbation-theory integrals are concentrated in precisely those functional classes of fields which allow a local gauge.

For non-Abelian fields, a gauge is locally well defined only if the fields are not too singular.^{1–3} The path integrals describing the quantum dynamics, in contrast, are concentrated in functions with poor differential properties. In this letter it is shown that this contradiction does not arise in perturbation theory, because of a UV regularization. If the latter is sufficient to remove the divergences, then among the sets of the complete measure for Gaussian integrals there are some which allow a local gauge. The exact coincidence of two functional boundaries of different types is interesting. This point has apparently not been mentioned previously in the literature, although an invariant regularization is an inseparable part of any attempt to rigorously define an integration over non-Abelian gauge fields.^{4–6}

Let us examine a regularization of the Yang–Mills Lagrangian by higher covariant derivatives:

$$L_{\text{YM}} \rightarrow L_{\text{YM}}^{\text{reg}} = \frac{1}{8} \text{tr} \left(F_{\mu\nu} F^{\mu\nu} + \frac{1}{\Lambda^{2m}} \nabla_{\kappa \dots \kappa} \nabla_{\lambda} F_{\mu\nu} \nabla^{\kappa \dots \kappa} \nabla^{\lambda} F^{\mu\nu} \right).$$

We use the Lorentz gauge, in which the regularized free propagator is

$$\mathcal{D}_{\mu\nu}^{ab}(p) = -\delta^{ab} \left(g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) \left(p^2 - \frac{1}{\Lambda^{2m}} p^{2+2m} \right)^{-1}. \quad (1)$$

The index of an arbitrary perturbation-theory diagram can be estimated from

$$\omega \leq 2 + 2m - (2 + 2m - D)\Pi - n_3 - 2n_4 - \dots - (2 + 2m)n_{4+2m}, \quad (2)$$

which generalizes a corresponding relation in Ref. 7, where $D=4$ and $m=2$. Here D is the dimensionality of the space-time, Π is the number of closed loops going into the diagram, and n_i is the number of vertices having i outgoing lines. It follows from estimate (2) that the condition under which all except single-loop divergences are removed is the inequality

$$m \geq D - 2. \quad (3)$$

The other diagrams are regularized by a modified Pauli–Willars procedure.⁷ If this inequality is strict, we obtain as a result a generating functional which has no UV divergences.

We denote by $(\cdot, \mathcal{D} \cdot)$ a correlation functional specified by propagator (1) in Euclidean space-time. As the initial range of definition of the corresponding Gaussian measure μ we can use the subspace defined by the condition $\partial^\nu A_\nu = 0$ in the space of moderate-growth distributions, S' . A measure exists on it by virtue of the Minlos theorem, and the following equality holds:

$$\exp\left\{-\frac{1}{2}(j, \mathcal{D} j)\right\} = \int_{S'} \exp\{i(A, j)\} \mu(dA_1). \quad (4)$$

To describe the properties of the carrier μ we use the notation of Ref. 8, where the case of a free boson measure was analyzed. Specifically, we assume $Q = 1 + x^2$, $P = 1 - \partial^2$. We denote by $H_{\alpha, \beta}$ the Hilbert space of transverse fields with the scalar product

$$\langle A, B \rangle = \int R A_\nu^a R B^{a\nu} dx, \quad \text{where } R = Q^{-\beta} P^\alpha. \quad (5)$$

The elements of $H_{\alpha, \beta}$ locally belong to Sobolev class L^2_q , where $q = 2\alpha$. We express the bilinear forms in (4) in terms of brackets (5), assuming $A \in H_{\alpha, \beta}$. We have the equality $(A, j) = \langle A, J \rangle$, where $J = (R^* R)^{-1} j$, and the conjugation is to be understood in the sense of a duality of S and S' . We also have $(j, \mathcal{D} j) = \langle T J, T J \rangle$, where $T = R^{-1} \mathcal{D}^{1/2} R^* R$. The measure μ is concentrated on $H_{\alpha, \beta}$ if the operator T belongs to the Hilbert-Schmidt class.⁹ Since R maps $H_{\alpha, \beta}$ onto L^2 isometrically, the operator T belongs to this class if and only if $R T R^{-1} = \mathcal{D}^{1/2} R^*$ is a Hilbert-Schmidt operator on L^2 . The kernel of the last integral operator is $K(x - y)(1 - y^2)^{-\beta}$, where

$$\tilde{K}_{\mu\nu}^{ab}(p) = \delta^{ab} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \left(p^2 + \frac{1}{\Lambda^{2m}} p^{2+2m} \right)^{-1/2} (1 + p^2)^\alpha,$$

and it is quadratically integrable if

$$\alpha < \frac{1}{2}(m+1) - \frac{D}{4}, \quad \beta > \frac{D}{4}. \quad (6)$$

The measure μ is thus concentrated on fields which are locally functions of the class L^2_q with any $q < m + 1 - D/2$. Spaces with indices

$$q \geq D/2 - 1 \quad (7)$$

are included here if and only if the condition of UV finiteness, $m > D - 2$, holds. This is exactly the limitation under which the gauge exists. The proof of this fact, given in Ref. 2, covers the case $D = 4$, $q \geq 1$, but it can be extended to other dimensionalities. We wish to stress that we can talk about only a local gauge here, because of Gribov copies.¹⁰ On the other hand, the existence of copies can be utilized to prove the need for limitation (7) by means of some simple scaling considerations [which, incidentally, also underlie estimate (2)].

We first wish to verify that smooth and rapidly decreasing copies exist. We assume $A_\nu(x) = a(r) \hat{n} \partial_\nu \hat{n}$, where $x \in R^3$, $\hat{n} = in_\nu \sigma_\nu$, $n_\nu = x_\nu / r$, and σ_ν are the Pauli matrices. The gauge transformation $g = \exp\{\alpha(r) \hat{n}\}$ returns this transverse field to the gauge surface if

$$\alpha'' + \frac{2}{r} \alpha' - \frac{1}{r^2} (2a + 1) \sin 2\alpha = 0. \quad (8)$$

Switching to the variable $t = \ln r$, and linearizing, we find that the roots of the characteristic equation are 1 and -2 . We denote by $\alpha(r)$ any smooth function which has an upper bound of $\pi/2$, no zeros, and the corresponding asymptotic behavior, e.g., $r/(1 - r^3)$. On it, we define $a(r)$ by means of Eq. (8). It is not difficult to verify that $a(r)$ has an r^2 behavior as $r \rightarrow 0$ and an r^{-3} behavior as $r \rightarrow \infty$. The field A and its copy A^g are thus regular at the origin and infinity, and they belong to any L_q^2 . We now perform the gauge transformation $A(x) \rightarrow \lambda A(\lambda x)$. This transformation does not violate the property of transversality. It sends gauge-equivalent fields back into equivalent fields coupled by the transformation $g(\lambda x)$. In the limit $\lambda \rightarrow \infty$ we have

$$\|\lambda A(\lambda x)\|_{L_q^2} \approx \lambda^{q+1-D/2} \|A(x)\|_{L_q^2}.$$

Accordingly, if (7) does not hold, this transformation can be used to bring the field and its copy arbitrarily close to zero in the sense of the L_q^2 topology. This was what we needed to demonstrate.

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