Topologically nontrivial nature of the universe in connection with the anisotropy of the background radiation

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There is a correlation between the anisotropy of the background radiation and a possible topologically nontrivial nature of the universe. The discovery of a large-scale (quadrupole) anisotropy is utilized to derive a new limitation on a possible topological "radius" of the universe in the theory of an inflationary universe. This limitation is $R > (2/e)R_H = 0.7\ R_H$, where R_H is the radius of the observable horizon.

The data obtained in measurements of the background radiation in the COBE project[1] support the assertion that an anisotropy of the background radiation has been discovered. The results of these measurements agree extremely well with the predictions of the existing theory of an inflationary universe (Refs. 2–4, for example). In the present letter, without getting into a discussion of the accuracy of the COBE observations, we show that the fact that there is a large-scale (quadrupole) anisotropy in the background radiation imposes a new constraint on a possible topological "radius" of the universe within an inflationary theory.

Let us briefly recall the model for the onset of an anisotropy in the background radiation according to the inflationary theories. It is convenient to assume that the inflation is generated by some scalar (inflanton) field φ with a nontrivial vacuum potential. Quantum fluctuations of this field, $\delta\varphi$, cause inhomogeneities in the energy density of the matter which arises; these inhomogeneities lead in turn to inhomogeneities of the metric. The latter inhomogeneities, surviving to the recombination era, serve as the reason for the appearance of inhomogeneities in the temperature of the background radiation. A relationship between the anisotropy of the background radiation and the quantum fluctuations $\delta\varphi$ of the inflanton field φ was found in Refs. 2–5:

$$\left(\frac{\Delta T}{T}\right)_l = \frac{K_l}{6\sqrt{l(l+1)}}\sqrt{\frac{\pi}{2}}\,\frac{H}{\dot{\varphi}}\,\delta\varphi(p)\,|_{p\sim H,} \tag{1}$$

where T is the temperature of the background radiation, l is the index of the harmonic in the multipole expansion of $\Delta T/T$ ($l \geqslant 2$), H is Hubble's constant, the dot over the φ means the time derivative, and $K_l \simeq 12/5$. By $\delta\varphi(p)$ we mean the contribution to $\delta\varphi$ from all perturbations in a unit interval of $\ln p/H$. The symbols $p \sim H$ mean that the values of $H/\dot{\varphi}$ are taken at the time at which p reaches a value on the order of H.

The fluctuations of a scalar field in a topologically trivial universe were found in, for example Refs. 6–8, where $\delta\varphi(p) = H/2\pi$ was derived. The independence of

$\delta\varphi(p)$ from p (a so-called flat spectrum) leads to the initial perturbations necessary for the formation of the observed large-scale structure.

We turn now to the onset of the large-scale anisotropy of background radiation in a universe whose space-time has a topologically nontrivial structure $M_n=(S^1)^n \times R^{4-n}$, where $n=1, 2, 3$; and S^1 means closure in terms of the spatial coordinate. We might note that the probability for the quantum generation of a universe with a structure of this sort is extremely large.[9] For simplicity we discuss a space-time with a flat metric (in other cases, the space-time becomes flat exponentially rapidly):

$$ds^2=dt^2-e^{2Ht}dl^2. \tag{2}$$

Let us find the quantum fluctuations of the field φ in a universe of this sort. The nontrivial nature of the topology reduces to conditions of the following type on the field:

$$\begin{aligned}&\varphi(x_0,x_1+n_1L_1,x_2+n_2L_2,x_3+n_3L_3)=\varphi(x_0,\mathbf{x}) \quad \text{for } M_3,\\ &\varphi(x_0,x_1,x_2+n_2L_2,x_3+n_3L_3)=\varphi(x_0,\mathbf{x}) \quad \text{for } M_2,\\ &\varphi(x_0,x_1,x_2,x_3+n_3L_3)=\varphi(x_0,\mathbf{x}) \quad \text{for } M_1,\end{aligned} \tag{3}$$

where $\mathbf{x}=(x_1, x_2, x_3)$, and L_1, L_2, L_3 are topological "radii" of the universe.

We have ignored the case in which the field φ can change sign as $x_i \to x_i+n_iL_i$, $i=1, 2, 3$ (the case of a "field with twisting"). We simply note, as was pointed out by Goncharov and Nesteruk,[10] that a twisted field cannot be an inflanton field. The reason is that the latter must have a nonvanishing classical condensate which is the same at all spatial points; this condition cannot be met if the field changes sign.

The equation for a scalar field with the metric in (2) takes the form

$$\ddot{\varphi}+3H\dot{\varphi}+m^2\varphi-e^{-2Ht}\nabla^2\varphi=0. \tag{4}$$

As an example we consider the M_3 case. The normalized solution of this equation is

$$\varphi=e^{i\mathbf{k}\cdot\mathbf{x}}\frac{H|\eta|^{3/2}}{(L_1L_2L_3)^{1/2}}\frac{\sqrt{\pi}}{2}[C_1H_\nu^{(1)}(k\eta)+C_2H_\nu^{(2)}(k\eta)], \tag{5}$$

where C_1 and C_2 are constants which satisfy $|C_2|^2-|C_1|^2=1$, $H_\nu^{(1,2)}$ are the Hankel functions of the first and second kinds, $\eta=-e^{-Ht}/H$ is the conformal time, $\mathbf{k}$ is the conformal momentum ($k=|\mathbf{k}|$), and $\nu^2=9/4-m^2/H^2$. Below we assume $m^2 \ll H^2$. This is the case of greatest interest in connection with the question of the anisotropy of the background radiation. Conditions (3) lead to a discrete $\mathbf{k}$ spectrum for the solution of (4):

$$\mathbf{k}^2=\sum_{s=1}^{3}\left(\frac{2\pi}{L_s}\right)^2 n_s^2\equiv k_n^2. \tag{6}$$

We now seek the fluctuations of a quantized scalar field of the form

$$\hat{\varphi}=\sum_n [a_{k_n}^{(-)}\varphi(t,\mathbf{x})+a_{k_n}^{(+)}\varphi^*(t,\mathbf{x})], \tag{7}$$

where $\alpha^{(-)}$ and $\alpha^{(+)}$ are annihilation and creation operators. The square of the fluctuations of such a field is

$$\langle\varphi^2\rangle \equiv \langle 0|\varphi^2|0\rangle = \sum_n{}' \varphi^*\varphi = \frac{H^2\eta^2}{L_1L_2L_3}\sum_n{}'\left(\frac{1}{2k_n}+\frac{1}{2k_n^3\eta^2}\right), \tag{8}$$

where $|0\rangle$ is the field vacuum; the summation is over all integer values of n_1, n_2, n_3; and the prime on the summation sign means that the term with $n_1=n_2=n_3=0$ is to be omitted. Here we are taking $|C_2|=1$ and $C_1=0$. These values are usually assumed in the case of a sufficiently prolonged inflation.

For simplicity we assume $L_i=L$, $i=1, 2, 3$. We go over to physical momenta $\mathbf{p}=\mathbf{k}e^{-Ht}$ and physical dimensions $R=Le^{Ht}$. Expression (8) then becomes

$$\langle\varphi^2\rangle = \frac{1}{R^3}\sum_n{}'\left[\frac{1}{2p_n}+\frac{H^2}{2p_n^3}\right], \tag{9}$$

where $p_n=|\mathbf{p}_n|$ is the magnitude of the physical momentum. Here

$$p_n^2 = \sum_{s=1}^{3}\left(\frac{2\pi}{R}\right)^2 n_s^2. \tag{10}$$

We turn to the renormalization of $\langle\varphi^2\rangle$. Taking the fluctuations to be perturbations against the background of a Minkowski vacuum, we subtract the term which corresponds to the contribution of the Minkowski vacuum. Doing this usually reduces to discarding the first term in (9). In a topologically nontrivial universe, however, a subtraction of the vacuum Minkowski term from the first term in (9) leads to a finite difference

$$\Delta = \frac{1}{R^3}\sum_n{}'\frac{1}{2p_n} - \frac{1}{(2\pi)^3}\int\frac{d^3p}{2p}. \tag{11}$$

It is not difficult to show by means of a regularization procedure using, say, the Einstein ζ function, that the difference Δ is $\Delta \approx -9/4\pi R^2$, so it falls off exponentially as the universe inflates. We will accordingly ignore the difference Δ below. What sort of fluctuations are generated by the second term in (9)? To renormalize this term we use the method proposed by Vilenkin,[11] who showed that it is sufficient to restrict the analysis to momenta $He^{-Ht} < p_n < H$ in a derivation of $\langle\varphi^2\rangle$ in the ordinary universe. In M_3 space, we can propose the following direct generalization of the formulas proposed by Vilenkin:[11]

$$\langle\varphi^2\rangle_{\text{ren}} = \frac{1}{R^3}\sum_{He^{-Ht}<p_n<H}\frac{H^2}{2p_n^3}. \tag{12}$$

Similar arguments lead to the following fluctuations of the field φ in an M_2 universe:

$$\langle\varphi^2\rangle_{\text{ren}} = \frac{1}{R^2}\sum_{He^{-Ht}<p_n<H}\int\frac{dp_1}{2\pi}\frac{H^2}{2p_{n_{2,3}}^3},$$
$$p_{n_{2,3}}^2 = p_1^2+(2\pi/R)^2(n_2^2+n_3^2). \tag{13}$$

For an M_1 universe we find

$$\langle\varphi^2\rangle_{\rm ren}=\frac{1}{R}\sum_{He^{-Ht}<p_n<H}\int\frac{dp_1dp_2}{(2\pi)^2}\frac{H^2}{2p_{n_3}{}^3},$$
$$p_{n_3}{}^2=p_1^2+p_2^2+(2\pi/R)^2(n_3^2). \tag{14}$$

As was mentioned above, to find the fluctuations $\delta\varphi(p)$ which appear in Eq. (1), we need to consider the contribution to $\sqrt{\langle\varphi^2\rangle_{\rm reg}}$ in (12)–(14) for all fluctuations in a unit interval of $\ln p/H$. In other words, we need to take the sum over momenta from p to pe ($e=2.17\ldots$) in (12)–(14). Here we should recall that we cannot go outside the limits of sums (12)–(14); i.e., $p>He^{-Ht}$ and $pe<H$.

It is easy to see from (12)–(14) that for all cases M_i, $i=1, 2, 3$, we can write

$$\delta\varphi(p)=\frac{H}{2\pi}f(p), \tag{15}$$

where $f(p)$ is a dimensionless function of the momentum. The case $f(p)=1$ corresponds to a trivial topology.

For M_1 we find an explicit expression for $f(p)$, once we evaluate the corresponding integrals:

$$f^2(p)=\frac{[x]}{x}-\frac{[xe]}{xe}+\psi([xe]+1)-\psi([x]+1), \tag{16}$$

where $x=pR/2\pi$, $\psi(y)$ is the psi (or digamma) function, and the square brackets mean the greatest integer.

It is not difficult to see that under the condition $xe<1$ we have $f(p)=0$. Later on, $f(p)$ increases, tending toward unity. Selecting a momentum p corresponding to the horizon scale (p_H), we see that under the condition $R<2\pi p_H^{-1}/e=(2/e)R_H$ (we have $R_H\sim 10^{28}$ cm today) there must be no large-scale anisotropy associated with the dimensions of the horizon (a quadruple anisotropy) at all.

The expressions for $f(p)$ in the M_2 and M_3 cases are much more complicated. Nevertheless, we have $f(p)=0$ at the same values of $R<2R_H/e$.

Since a quadrupole anisotropy of the background radiation was detected in the COBE experiment, it can be concluded that a topological radius of the universe

$$R>\frac{2}{e}R_H\approx 0.7\,R_H \tag{17}$$

is possible. The best limitations presently available[12, 13] are $R>R_H/7.5$.

We note in conclusion that in deriving limitation (17) we made use of just the fact that there is a quadrupole anisotropy. If experiments currently in the planning stage (COBE and RELIKT-2) allow us to draw a conclusion about the nature of the background radiation spectrum, we will be in a position to substantially strengthen limitation (17). If a deviation from a flat spectrum is observed, that result might be interpreted as indicating a possible topological nontrivial nature of the universe at large scales. Furthermore, by analyzing the anomalies in the background-radiation spectrum at scales smaller than the horizon one can extract information on the topological structure of the universe at smaller scales also.

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