New nonlinear equations for low-frequency electromagnetic waves in nonuniform high-beta plasmas

P. K. Shukla and L. Stenflo
Institut für Theoretische Physik IV, Ruhr-Universität Bochum, D-44780 Bochum, Germany
**Department of Plasma Physics, Umeå University, S-90187 Umeå, Sweden*

(Submitted 22 April 1993)
Pis'ma Zh. Eksp. Teor. Fiz. **57**, No. 11, 680–683 (10 June 1993)

A new set of nonlinear fluid equations governing low-frequency electromagnetic turbulence is derived for a nonuniform high-beta plasma.

More than fifteen years ago, Strauss[1] derived a pair of coupled nonlinear equations for low-frequency (in comparison with the ion gyrofrequency) electromagnetic waves in a uniform low-β, plasma ($\beta=8\pi n_0 T/B_0^2 \ll 1$, where n_0 is the unperturbed particle number density, T is the plasma temperature, and B_0 is the strength of the external magnetic field). The Strauss equations, which govern the nonlinear dynamics of nondispersive Alfvén waves, have been derived from the MHD system of equations, assuming that the plasma is incompressible and that the transverse(to the external magnetic field) component of the wave magnetic field is much larger than the compressional magnetic field perturbation. The Strauss equations have been generalized to include the effect of the plasma compressibility, the parallel electron inertial force, and the plasma nonuniformities.[2–7] Here one encounters linear and nonlinear couplings between the electron drift and dispersive shear Alfvén waves. The nonlinear equations are useful for studying a variety of problems, including low-frequency turbulence and self-organization processes in magnetized plasmas.

To the best of our knowledge, other authors have not derived the nonlinear equations for electromagnetic waves in high-beta (i.e., $\beta \sim 1$) plasmas. In such plasmas, one must incorporate the combined effect of the sheared and the compressional magnetic field perturbations. Our objective here is to present a set of model nonlinear fluid equations that govern the dynamics of weakly interacting low-frequency electromagnetic modes in high-β collisionless plasmas which contain an equilibrium density gradient.

Let us consider the nonlinear propagation of finite amplitude electromagnetic waves in a nonuniform high-β plasma consisting of electrons and ions. In a Cartesian coordinate system, we assume that the external magnetic field $\mathbf{B}_0$ is directed along the x axis, and that the equilibrium density gradient $(d_x n_0)$ is directed along the x axis. In the presence of low-frequency (in comparison with $\omega_{ci}=eB_0/m_i c$, where e is the electron charge, m_i is the ion mass, and c is the speed of light) electromagnetic fields, the perpendicular components of the electron and ion fluid velocities are, respectively,

$$\mathbf{v}_{e\perp} \approx \mathbf{v}_E+\mathbf{v}_D+v_{ez}\frac{\mathbf{B}_{1\perp}}{B_0}-\frac{B_{1z}}{B_0}(\mathbf{v}_E+\mathbf{v}_D), \tag{1}$$

and

$$\mathbf{v}_{i\perp} \approx \mathbf{v}_E + \mathbf{v}_p + v_{iz}\frac{\mathbf{B}_{i\perp}}{B_0} - \frac{B_{1z}}{B_0}\mathbf{v}_E, \tag{2}$$

where $\mathbf{v}_E = (c/B_0)\mathbf{E}\times\hat{z}$, $\mathbf{v}_D = -(cm_e v_{te}^2/eB_0)\hat{\mathbf{z}}\times\nabla\ln n_e$, $\mathbf{v}_p = (c/B_0\omega_{ci})(\partial_t + \mathbf{v}_E \times\nabla_\perp + v_{iz}\partial_z)\mathbf{E}_\perp$, $\hat{\mathbf{z}}$ is the unit vector along the external magnetic field, $\mathbf{E}_\perp$ is the perpendicular component of the wave electric field vector, n_e is the total electron number density, $v_{te} = (T_e/m_e)^{1/2}$ is the electron thermal velocity, T_e is the electron temperature, and $\mathbf{B}_1 = \mathbf{B}_{1\perp} + \hat{\mathbf{z}}B_{1z}$ is the perturbed magnetic field. For simplicity, the ion temperature is assumed to be much smaller than T_e. The parallel components of the electron (v_{ez}) and ion (v_{iz}) fluid velocities are determined from the corresponding momentum balance equations.

For our purpose, we need the electron continuity equation

$$\partial_t n_e + \nabla\cdot(n_e\mathbf{v}_e) = 0, \tag{3}$$

the conservation of the total current density in the quasi-neutral approximation $n_e = n_i = n$ [which is valid for a dense plasma with $\omega_{pi} = (4\pi n_0 e^2/m_i)^{1/2} \gg \omega_{ci}$]

$$\nabla\cdot\mathbf{j} \equiv e\nabla\cdot[n(\mathbf{v}_i - \mathbf{v}_e)] = 0, \tag{4}$$

and Faraday and Ampere's laws

$$\partial_t\mathbf{B}_1 = -c\nabla\times\mathbf{E}, \tag{5}$$

and

$$\nabla\times\mathbf{B}_1 = \frac{4\pi}{c}\mathbf{j}, \tag{6}$$

where the subscript $e(i)$ stands for the electrons (ions). Furthermore, we have the displacement current in (6), since we are dealing with low phase velocity (in comparison with c) electromagnetic waves.

The nonlinear model equations for finite amplitude waves can now be derived. We introduce $n_1 = n - n_0$, where $n_1(\ll n_0)$ is the number density perturbation. Substituting (1) into (3) and using (6), we obtain

$$(\partial_t + \mathbf{v}_E\cdot\nabla)[\ln n_0(x) + N - b] + d_z\left[v_{iz} - \frac{c}{4\pi e n_0}(\nabla\times\mathbf{B}_1)_z\right] + \partial_t\frac{b^2}{2} = 0, \tag{7}$$

where $N = n_1/n_0$, $b = B_{1z}/B_0$, $\partial_z = d_z + (\mathbf{B}_{1\perp}/B_0)\cdot\nabla$, and v_{iz} is assumed to be determined by

$$(\partial_t + \mathbf{v}_E\cdot\nabla)v_{iz} = \frac{e}{m_i}E_z + \frac{e}{m_i c}(\mathbf{v}_E\times\mathbf{B}_1)_z, \tag{8}$$

where E_z is the parallel component of the wave electric field. Here we have assumed that $v_{ez,iz}\partial_z \ll \mathbf{v}_E\cdot\nabla$.

Inserting (1) and (2) into (4) and using (6), we have

$$\partial_t\nabla\cdot\mathbf{E}_\perp + \nabla\cdot(\mathbf{v}_E\cdot\nabla\mathbf{E}_\perp) + \frac{v_A^2}{c}d_z(\nabla\times\mathbf{B}_1)_z - \frac{B_0 c_s^2}{c}(\hat{\mathbf{z}}\times\nabla N)\cdot\nabla b = 0, \tag{9}$$

where $v_A = B_0/(4\pi_0 m_i)^{1/2}$, and $c_s = (T_e/m_i)^{1/2}$ are the Alfvén and ion-acoustic velocities, respectively.

Subtracting the parallel component of the ion momentum equation from the corresponding electron momentum equation and making use of (6), we obtain

$$(\partial_t + \mathbf{v}_E \cdot \nabla)(\nabla \times \mathbf{B}_1)_z = \frac{\omega_{pe}^2}{c}\left[E_z + \frac{m_e v_{te}^2}{e} d_z[\ln n_0(x) + N] + \mathbf{E}_\perp \cdot \frac{\mathbf{B}_1}{B_0}\right], \tag{10}$$

where $\omega_{pe} = (4\pi n_0 e^2/m_e)^{1/2}$ is the electron plasma frequency.

The compressional magnetic field perturbation B_{1z} is determined by taking the curl of (6) and using (1) and (2), and then considering the z component of the resulting equation. The result is

$$(\partial_t^2 - v_A^2 \nabla^2) b = c_s^2 (1-b) \nabla_\perp^2 N + \frac{c}{B_0}[\nabla \times (\mathbf{v}_E \cdot \nabla \mathbf{E}_\perp)]_z + \frac{v_A^2}{B_0^2}[\nabla \times (\nabla \times \mathbf{B}_1)_z \mathbf{B}_1]_z. \tag{11}$$

Here we have assumed that the wavelengths are much smaller than the density inhomogeneity scale length. The magnetic field perturbation $\mathbf{B}_1$ appearing in (7)–(11) is expressed in terms of E by means of (5). The latter also gives

$$\partial_t(\nabla \times \mathbf{B}_1)_z = c(\nabla_\perp^2 E_z - \partial_z \nabla \cdot \mathbf{E}_\perp). \tag{12}$$

We have thus derived five coupled nonlinear equations governing the evolution of the five physical variables n_1, E_x, E_y, E_z, and v_{iz}. These variables generalize the preceding equations[2–7] by including the compressional wave magnetic fields along the external magnetic field lines.

Let us now consider a well-known limiting case. When $\mathbf{E} = -\nabla\phi - (\hat{z}/c)\partial_t A_x$, $\mathbf{B}_1 = \nabla A_z \times \hat{\mathbf{z}}$, $B_{1z} = 0$, and $v_{iz} = 0$, where ϕ is the scalar potential and A_z is the parallel component of the vector potential, the SS equations (7)–(11) reduce to[2–7]

$$d_t[\ln n_0(x) + N] + \frac{c}{4\pi e n_0} d_z \nabla_\perp^2 A_z = 0, \tag{13}$$

$$d_t \nabla_\perp^2 \phi + \frac{v_A^2}{c} d_z \nabla_\perp^2 A_z = 0, \tag{14}$$

and

$$d_t[(1 - \lambda_e^2 \nabla_\perp^2) A_z] + c\partial_z\phi - \frac{c m_e v_{te}^2}{e} d_z[\ln n_0(x) + N] = 0, \tag{15}$$

where $\lambda_e = c/\omega_{pe}$ is the collisionless electron skin depth, $d_t = \partial_t + (c/B_0)(\hat{\mathbf{z}} \times \nabla\phi) \cdot \nabla$, and $d_z = \vartheta_z + (1/B_0)(\nabla A_z \times \hat{\mathbf{z}}) \cdot \nabla$. In the absence of the density inhomogeneity and the parallel components of the electron inertial and pressure gradient forces, Eq. (15) takes the form

$$d_t A_z + c\partial_z \phi = 0. \tag{16}$$

Equations (14) and (16) are evidently the Strauss equations.[1]

To summarize, we have presented for the first time a set of coupled nonlinear equations that govern the dynamics of finite amplitude low-frequency electromagnetic fluctuations in nonuniform high-β plasmas. In the linear limit these equations give a local dispersion relation which exhibits coupling between the magnetosonic, shear-Alfvén, and electron drift waves. The present nonlinear equations should be useful for studies of instabilities, turbulence, and vortices involving electromagnetic perturbations in high-β plasmas.

This research was supported by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 191 "Physikalische Grundlagen der Niedertemperaturplasmen" and the Commission of the European Economic Community through the SCIENCE (Twinning and operations) program under the Contract No. SC1-CT92-0773.

[1] H. R. Strauss, Phys. Fluids **19**, 134 (1976).
[2] A. Hasegawa and M. Wakatani, Phys. Fluids **26**, 2770 (1983).
[3] H. U. Rahman and J. Weiland, Phys. Rev. A **28**, 1678 (1983).
[4] P. K. Shukla and M. Y. Yu, SFB Report L1-110 (1983), Ruhr-University Bochum, Germany.
[5] P. K. Shukla, M. Y. Yu, H. U. Rahman, and K. H. Spatschek, Phys. Rep. **105**, 227 (1984).
[6] B. B. Kadomtsev and O. P. Pogutse, JETP Lett. **39**, 269 (1984).
[7] V. Petviashvili and O. Pokhotelov, *Solitary Waves in Plasmas and in the Atmosphere*, Gordon and Breach, New York, 1992.

Submitted in English by the authors