# Dual flip-flop: A new paradigm of collective dynamics

V. B. Andreichenko[1)]
*Landau Institute for Theoretical Physics, GSP-1 117940 Moscow V-334, Russia*

(Submitted 11 May 1993)
Pis'ma Zh. Eksp. Teor. Fiz. **57**, No. 11, 732–736 (10 June 1993)

We define new nonlocal dynamics as alternating inversions of the duality of the spin configurations on an expanded space of order–disorder variables. Some components of the novel procedure of the duality inversion are known, but the principal "absolutely parallel" operation of the dual complementation is new. The fixing of the duality reduces our dynamics to the Swendsen–Wang cluster algorithm.

Dynamic properties which are sufficiently close to the second-order phase transition, like thermodynamic properties, can be described by a set of critical exponents. The singularities of the thermodynamic quantities at the transition point are determined by the structure of the Hamiltonian and the number of components of the order parameter.[1] The hypothesis of the scale invariance makes it possible to obtain certain relations between the critical exponents but is not sufficient to completely define the dynamic exponents.[1,2] Even in the 2D Ising model, for which exact values of the complete set of the thermodynamic exponents are known, the value of the dynamic exponent[3] $z=2.183\pm0.005$, which determines the homogeneous relaxation time $\tau\sim|T-T_c|^{-z}$, is known only as a result of the numerical simulations with local dynamic algorithms. The fact that $z$ is close to 2, with the value of the exponent $\nu=1$ of the correlation length $\xi\sim|T-T_c|^{-\nu}$, ensues from the local character of the dynamics. Here, like in the random walk problem, to cover the distance $\xi$ requires a time proportional to $\xi^2$.

Different mechanisms ensuring relaxation can lead to diversity in critical dynamic behavior.[2] Near the critical point the correlation length becomes very large, and lattice spins cannot adequately describe relevant degrees of freedom. The nonuniversality of the critical dynamics has been demonstrated by Swendsen and Wang[4] (SW), who have proposed the relaxation mechanism of a nonlocal type. The SW dynamics, which operates with spin clusters, yields the value of the critical dynamic exponent[4] $z\approx0.35$. Clusters in the SW conjecture seem to be closer to the genuine collective degrees of freedom, correctly describing the spatial and time correlations. However, the possibility of even a faster relaxation is still open to discussion.[5,6]

The 2D Ising model is equivalent to a model of free fermions,[7] which are known as the proper variables for describing static properties, whereas SW clusters are good objects for fast dynamics. The introduction of fermions is connected with duality transformation. There arises, therefore, a natural question about the relationship between clusters, dual symmetry, and fermions.

In what follows we shall study implications of the dual symmetry in the cluster description, and vice versa, the role of randomly generated clusters in the duality transformation. The solution of this problem will make it possible to construct a system of collective degrees of freedom which describe the thermodynamics and dynamics in the critical region. In the present paper we propose a new principle of relaxation for the 2D Ising model and point to possible ways of finding generalizations for other systems.

The 2D Ising model, which is defined by the partition function

$$Z(\beta)=\sum_{\{\sigma\}}\prod_{x,\alpha}e^{\beta J\sigma_x\sigma_{x+\alpha}}, \tag{1}$$

can be mapped by the duality transformation on itself[8] but at a different temperature $T^*$, which monotonically decreases with increasing temperature $T$ of the original model,

$$e^{-2\beta^* J}=\tanh\beta J. \tag{2}$$

Here $\beta$ and $\beta^*$ are inverse temperatures of mutually dual spin systems $\{\sigma\}$ and $\{\mu\}$, where $\{x\}$ are the sites and $\{\alpha\}$ are the two basic vectors of the square lattice considered below. The transformation to dual spin variables $\{\mu\}$ is realized after the summation over the spins of the original lattice $\{\sigma\}$ is performed.[9] The following form of the exponent in the partition function (1)

$$e^{\beta J\sigma_x\sigma_{x+\alpha}}=\cosh\beta J(1+\sigma_x\sigma_{x+\alpha}\tanh\beta J)$$

is employed.

In the transformation to noninteracting clusters the summation over spins of the original lattice is preceded by a different representation[10] of the same exponent

$$e^{\beta J\sigma_x\sigma_{x+\alpha}}=e^{\beta J}(e^{-2\beta J}+(1-e^{-2\beta J})\delta_{\sigma_x,\sigma_{x+\alpha}}).$$

The clusters occurring in the result consist of bonds placed on links of the lattice with the probability

$$p_{x,\alpha}=1-e^{-\beta J(\sigma_x\sigma_{x+\alpha}+1)}. \tag{3}$$

To unify the duality transformation and noninteracting clusters, it is sufficient to study implications of the dual symmetry of the mixed four-spin correlation function

$$Q=\langle\sigma_x\sigma_{x+\alpha}\mu_x\mu_{x+\alpha}\rangle_\sigma=\langle\sigma_x\sigma_{x+\alpha}e^{-2\beta J\sigma_x\sigma_{x+\alpha}}\rangle_\sigma$$

in its representation via parameters of the clusters. Here the neighboring couples of spins are positioned at the ends of the intersecting dual links. In this case the subscript $\sigma$ means the procedure of averaging

$$\langle(...)\rangle_\sigma=\frac{\Sigma\{\sigma\}e^{-\beta H\{\sigma\}}(...)}{\Sigma\{\sigma\}e^{-\beta H\{\sigma\}}}$$

over the original spin lattice having the temperature $\beta$. For the disorder operators we use their representations[11,12]

$$\mu_x=\prod_{-\infty}^{x}e^{-2\beta J\sigma\sigma'} \tag{4}$$

in terms of the product of exponents along the contour at the dual lattice.

Expanding the exponent in the four-spin neighboring-cross-linked correlation function $Q$, we can express it in terms of the pair correlation function $G=\langle\sigma_x\sigma_{x+\alpha}\rangle_\sigma$ of the original lattice spins

$$Q=G\cosh 2\beta J-\sinh 2\beta J.$$

An analogous expression through the pair correlation function of the dual spins

$$F=\langle\mu_x\mu_{x+\alpha}\rangle_\sigma=\langle e^{-2\beta J\sigma_x\sigma_{x+\alpha}}\rangle_\sigma$$

looks especially simple if the dual temperature is introduced:

$$Q=\sinh 2\beta^* J-F\cosh 2\beta^* J.$$

The respective expressions for the correlation functions, which pertain to the dual lattice

$$G^*=\langle\mu_x\mu_{x+\alpha}\rangle_\mu,$$

$$F^*=\langle\sigma_x\sigma_{x+\alpha}\rangle_\mu=\langle e^{-2\beta^* J\mu_x\mu_{x+\alpha}}\rangle_\mu,$$

$$Q^*=\langle\sigma_x\sigma_{x+\alpha}\mu_x\mu_{x+\alpha}\rangle_\mu=\langle\mu_x\mu_{x+\alpha}e^{-2\beta^* J\mu_x\mu_{x+\alpha}}\rangle_\mu,$$

are obtained through the permutation $\beta\to\beta^*$, $\beta^*\to\beta$. Taking into account the well-known consequences $G^*=F$ and $F^*=G$ of the dual symmetry,[11] enable one at dual temperatures to relate the pair correlation functions of order and disorder variables, it is easy to show the anti-self-duality of the mixed four-spin correlator $Q$:

$$Q^*=-Q. \tag{5}$$

To clarify the consequences of the dual symmetry in terms of the lattice links, we introduce the probabilities $\omega_\pm$ of the same (opposite) orientation of spins at the neighboring sites ($\omega_+ + \omega_- = 1$). The mixed four-spin correlator $Q$ can be represented, in terms of the probabilities $\omega_\pm$

$$Q=e^{-2\beta J}\omega_+ - e^{2\beta J}\omega_-.$$

From the anti-self-duality (5) of $Q$ we derive the relationships between the probabilities $\omega_\pm$ at the dual links

$$\omega_+^*=\cosh\beta J(e^{-\beta J}\omega_+ + e^{\beta J}\omega_-),$$

$$\omega_-^*=\sinh\beta J(e^{-\beta J}\omega_+ - e^{\beta J}\omega_-).$$

Note the substantial temperature dependence which can be cancelled in the cluster representation.

Let $\omega_1$ ($\omega_0=1-\omega_1$) be the probability for the presence (absence) of the cluster bond at the link of the lattice $\{\sigma\}$. The relationship between the probabilities $\omega_{1,0}$ and $\omega_\pm$ ensues from (3)

$$\omega_1=(1-e^{-2\beta J})\omega_+,$$

$$\omega_0=e^{-2\beta J}\omega_+ + \omega_-.$$

The consequence of the anti-self-duality (5) of the cross-linked correlator $Q$ in this representation is a very simple relationship between the probabilities for finding a cluster bond at the intersecting links of the original and dual lattices:

$$\omega_1+\omega_1^*=1, \quad \omega_0+\omega_0^*=1. \tag{6}$$

We thus obtain a symmetric description of clusters at the original and dual lattices. Here the temperature dependence reveals itself only implicitly. At low temperatures bonds are localized largely at the original lattice, while at high temperatures they move to the dual lattice. Let us connect one cluster bond with each intersection of links of the original and dual lattices. This bond can be positioned only at one of these two links. This simple scheme automatically ensures the dual symmetry (6) of the probabilities $\omega_{1,0}$. An unambiguous correspondence of clusters at the dual lattices can be logically called dual complementation.

The absence of an unambiguous correspondence between spin configurations of the original and dual lattices is quite natural. Like in the conventional Fourier transformation, the configurations at the dual lattice are the result of the summation performed over all configurations at the original lattice. However, it is possible, using the dual complementation, to establish a correlation between spin configurations after mutually dual lattices.

First, we should build a configuration of bonds for the given spin configuration by means of probabilities $p_{x,\alpha}$ (3). Then by the dual complementation construct a configuration of bonds at the dual lattice. We must restore tn this case the spin configuration for the given bond configuration. For this purpose we can randomly choose the common orientation of spins of each cluster.

We call this construction the duality inversion of the spin configuration. Note its stochastic character. Here the unambiguous procedure of the dual complementation is "wrapped" on both sides with a probablistic correspondence between the spin and bond configurations.

Consequently, there naturally arises a nonlocal method for the renewal of spins, which can actually be described as alternating inversions of the duality of spin configurations on an expanded space of the order-disorder variables. The proper variables to describe the static properties (fermions) are defined in the same space as the products of these variables.[11] The arising nonlocal dynamics can be called dual flip-flop (DFF). The total cycle consists of two duality inversions. To "cut" the DFF cycle into operations of the dual complementation, we shall obtain two SW algorithms.

The duality transformation plays the same role as the Fourier transformation. The argument in favor of the DFF dynamics is the success obtained in the removal of the critical slowing down in the Fourier acceleration algorithm,[13,14] which actually is alternating the application of the Monte Carlo methods in the coordinate space and momentum space. It is clear that the only straightforward way to verify the absence of the critical slowing down is to perform simulations which are now in progress.

The diversity of perfect and inhomogeneous systems, which possess the dual symmetry,[9,15] is a natural area for DFF applications. In the context of the hardware implementation, it is worth noting the "absolutely parallel" character of the dual complementation procedure. The possibility of solving other problems pertaining to cluster algorithms at the hardware implementation has been demonstrated by a new specialized processor[16,17] which employs the Wolff one-cluster algorithm.[18]

We wish to thank V. Dotsenko, A. Shabat, and L. Shchur for valuable discussions. We also thank A. Talapov for many discussions and for a critical reading of the manuscript.

This work was supported, in part, by a Sloan Foundation Grant awarded by the American Physical Society.

[1)]e-mail: vovaitp.sherna.msk.su

---

[1] L. D. Landau and E. M. Lifshits, *Statistical Physics*, Moscow, 'Nauka,' 1976.
[2] E. M. Lifshits and L. P. Pitaevsky, *Physical Kinetics*, Moscow, 'Nauka,' 1979.
[3] B. Dammann and J. D. Reger, Europhys. Lett. **21**, 157 (1993).
[4] R. H. Swendsen and J.-S. Wang, Phys. Rev. Lett. **58**, 86 (1987).
[5] X. J. Li and A. D. Sokal, Phys. Rev. Lett. **63**, 827 (1989); Phys. Rev. Lett. **67**, 1482 (1991).
[6] D. W. Heermann and A. N. Burkitt, Physica A **162**, 210 (1990).
[7] T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).
[8] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252 (1941).
[9] R. Savit, Rev. Mod. Phys. **52**, 453 (1980).
[10] P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Jpn. Suppl. **26**, 11 (1969); C. M. Fortuin, P. W. Kasteleyn, Physica (Utrecht) **57**, 536 (1972).
[11] Vic S. Dotsenko and Vl. S. Dotsenko, Adv. Phys. **32**, 129 (1983).
[12] L. P. Kadanoff and H. Ceva, Phys. Rev. B **3**, 3918 (1971).
[13] G. G. Batrouni, G. R. Katz, A. S. Kronfeld *et al.*, Phys. Rev. D **32**, 2736 (1985).
[14] J. B. Kogut, Nucl. Phys. B **275** [FS17], 1 (1986).
[15] H. R. Jauslin and R. H. Swendsen, Phys. Rev. B **24**, 313 (1981).
[16] A. L. Talapov, L. N. Shchur, V. B. Andreichenko, and Vl. S. Dotsenko, Mod. Phys. Lett. B **6**, 1111 (1992).
[17] A. L. Talapov, V. B. Andreichenko, Vl. S. Dotsenko, and L. N. Shchur, to be published in Int. J. Mod. Phys. C.
[18] U. Wolff, Phys. Rev. Lett. **62**, 361 (1989).

Submitted in English by the author