Antibrackets and localization of (path) integrals

A. P. Nersessian
Laboratory of Theoretical Physics, JINR, Dubna 141980

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The transparent method for the invariant (Hamiltonian) description of equivariant localization of the integrals over phase space is proposed. This method uses the odd symplectic structure, which is constructed over tangent bundle of the phase space and which permits straightforward generalization of the path integrals. A method of supersymmetrization of a wide class of Hamiltonian systems has also been developed.

1. A number of papers (for example, Refs. 1–3), in which exact evaluation of the phase space path integrals was studied using corresponding generalization[1] of the Duistermaat–Heckman localization formula[4] (the DH formula), has recently been published. According to this formula, if on the compact manifold M provided with the symplectic structure $\omega=\frac{1}{2}\omega_{ij}dx^i\wedge dx^j$ the Hamiltonian $H(x)$ defines the action of the group $U(1)\sim S^1$, then

$$Z_0=\int_M e^H(\omega)^N=\sum_{dH=0}\frac{e^H\sqrt{\det\omega_{ij}}}{\sqrt{\det(\partial^2H)/(\partial x^i\partial x^j)}}. \tag{1}$$

Using its path integral generalization, we can localize the phase space path integral into (finite-dimensional) integral over classical phase space.

This approach turned out to be useful for a number of problems,[2] particularly in topological field theories. It formed the basis for a conceptually new method of describing supersymmetric theories.[3]

In the present letter we propose a simple method for the invariant description of the DH localization. Following[1-3] we write the integral (1) in the form

$$Z_0=\int_M e^{H(x)}\det w_{ij}d^{2N}x=\int_{\mathcal{M}} e^{H-F}d^{2N}xd^{2N}\theta, \tag{2}$$

where θ^i are auxiliary Grassmannian fields $p(\theta^i)=p(x^i)+1$, which correspond to 1-forms dx^i, $\mathcal{M}$ is a supermanifold associated with the tangent bundle of M [$z^A=(x^i,\theta^i)$ are the local coordinates on M], and

$$F(z)=-\tfrac{1}{2}\theta^i\omega_{ij}\theta^j. \tag{3}$$

After that we shall define on $\mathcal{M}$ the *odd symplectic structure.* The corresponding odd Poisson brackets (antibrackets) give the Hamiltonian description (and natural interpretation) of the DH localization, without introduction of the additional structures which are used in the cited papers.

We show, in addition, that the use of antibrackets gives a simple supersymmetrization method for the Hamiltonian systems, which define the isometries of the Riemannian metric on their phase space.

Finally, these constructions can be generalized straightforwardly to this case if M is a symplectic *supermanifold.* Moreover, they are completely symmetrical according to the relation of the initial and auxiliary coordinates.

All constructions presented here relate to the finite-dimensional integrals over compact symplectic manifolds. One can accomplish their generalization for the path integrals by lifting them to the loop space by analogy with the procedure in Refs. 1–3. It does not fundamentally change this description scheme.

Notice that this method is naturally connected with the Batalin–Vilkovisky quantization formalism.[5]

2. Let us provide the supermanifold $\mathscr{M}$, which we defined above, with odd symplectic structure,

$$\Omega_1=\omega_{ij}dx^i\wedge d\theta^j+\omega_{ij,k}\theta^j dx^i\wedge dx^k, \tag{4}$$

where ω_{ij} corresponds to the symplectic structure on M.

The odd Poisson brackets (antibrackets), corresponding to (4)

$$\{f,g\}_1=\frac{\partial_r f}{\partial z^A}\Omega_1^{AB}\frac{\partial g}{\partial z^B}, \tag{5}$$

are defined by the conditions

$$\{x^i,x^j\}_1=0,\quad \{x^i,\theta^j\}_1=-\{\theta^j,x^i\}_1=\omega^{ij},\quad \{\theta^i,\theta^j\}_1=-\{\theta^j,\theta^i\}_1=\frac{\partial\omega^{ij}}{\partial x^k}\theta^k, \tag{6}$$

where $\omega^{ij}\omega_{jk}=\delta^i_k$. The antibrackets (5) and (6) satisfy the Jacobi identity:

$$(-1)^{[p(f)+1][p(h)+1]}\{f,\{g,h\}_1\}_1+\text{cycl. perm. (f,g,h)}=0. \tag{7}$$

Let us map the functions on M into the odd functions on $\mathscr{M}$:

$$f(x)\to Q_f(z)=\{f(x),F(z)\}_1,$$

where F is defined by expression (3). It puts the Hamiltonian dynamics $[H(x),\omega,M]$ into the odd Hamiltonian dynamics $(Q,\Omega_1,\mathscr{M})$, where

$$Q=\{H,F\}_1, \tag{8}$$

with the equation of motion

$$\frac{dx^i}{dt}=\{x^i,Q\}_1=\{x^i,H_0\}_0\equiv\xi^i_H,\quad \frac{d\theta^i}{dt}=\{\theta^i,Q\}_1=\frac{\partial\xi^i_H}{\partial x^j}\theta^j. \tag{9}$$

This dynamics is supersymmetric: From the closure of ω follows $\{F,F\}_1=0$, and using (8), we obtain a simple superalgebra

$$\{H\pm F,H\pm F\}_1=\pm 2Q,$$

$$\{H+F,H-F\}_1=\{H\pm F,Q\}_1=\{Q,Q\}_1=0. \tag{10}$$

The following correspondence is obvious:

$$\{H,\ \ \}_1=\xi^i_H\frac{\partial}{\partial\theta^i}\to i_H\text{— the operator of interior product on } \xi_H;$$

$$\{F,\ \ \}_1=\theta^i\frac{\partial}{\partial x^i}\to d\text{— the operator of exterior differentiation;}$$

$$\{Q,\ \ \}_1=\theta^i_H\frac{\partial}{\partial x^i}+\xi^i_{H,k}\theta^k\frac{\partial}{\partial\theta^i}\to\mathscr{L}_H\text{—the Lie derivative along } \xi_H. \tag{11}$$

Taking into account the Jacobi identity (7), we have

$$\{H,F\}_1=Q\to di_H+i_Hd=\mathscr{L}_H\text{—homotopy formula.}$$

As we can see, the supersymmetry of $(Q,\Omega_1,\mathscr{M})$ corresponds to the equivalent differentiation $d_H=d+i_H$.

Following Refs. 1 and 2, we assume that on M the Riemannian metric g_{ij}, which is Lie-derived with ξ^i_H, is defined also. Then the odd function

$$\tilde{Q}=\xi^i_Hg_{ij}\theta^j\equiv\xi_i\theta^i \tag{12}$$

is the integral of motion of (9):

$$\mathscr{L}_Hg=0\to\{Q,\tilde{Q}\}_1=0. \tag{13}$$

We also have

$$\{F,\tilde{Q}\}_1=-F_2,\quad \{H,\tilde{Q}\}_1=H_2,$$

where

$$H_2=\xi^i_Hg_{ij}\xi^j_H,\quad F_2=\tfrac{1}{2}\,\theta^i\omega_{(2)ij}\theta^j,\quad \omega_{(2)ij}=\frac{\partial\xi_i}{\partial x^j}-\frac{\partial\xi_j}{\partial x^i}\,.$$

3. Now we shall demonstrate the derivation of the DH formula (1) using the constructions presented above.

Let us consider the integral

$$Z_\lambda=\int_{\mathscr{M}}\exp(H-F-\lambda\{H+F,\tilde{Q}\}_1)d^{4N}z, \tag{14}$$

where λ is an arbitrary numerical parameter.

As in Sec. 1, we assume that $\mathscr{M}$ is associated with the tangent bundle of the compact symplectic manifold M, and define on it the odd symplectic structure (4). We also assume that the Hamiltonian $H(x)$ defines on M the action of $U(1)\sim S^1$, that M is provided with the Riemannian structure g_{ij}, which is Lie-derived with ξ^i_H, and that F and $\tilde{Q}$ are defined by expressions (3) and (12).

The vector fields (11) preserve the "volume form" $d^{4N}z=d^{2N}xd^{2N}\theta$. From (10) and (13) we deduce

$$\{H+F,e^{H-F-\lambda\{H+F,\tilde{Q}\}_1}\}_1=0,\quad \{Q,e^{H-F-\lambda\{H+F,\tilde{Q}\}_1}\}_1=0.$$

Therefore, the integral (14) is invariant under equivariant and Lie transformations along ξ^i_H. We have

$$\{Q,\tilde{Q}e^{H-F-\lambda\{H+F,\tilde{Q}\}_1}\}_1=0.$$

Using these expressions and the fact that the integral of an equivariantly exact form vanishes, we obtain

$$\frac{dZ_\lambda}{d\lambda}=\int_{\mathcal{M}}\{H+F,\tilde{Q}\}_1 e^{H-F-\lambda\{H+F,\tilde{Q}\}_1}d^{4N}z$$

$$=\int_{\mathcal{M}}\{H+F,\tilde{Q}e^{H-F-\lambda\{H+F,\tilde{Q}\}_1}\}_1 d^{4N}z$$

$$-\int_{\mathcal{M}}\tilde{Q}\{H+F,e^{H-F-\lambda\{H+F,\tilde{Q}\}_1}\}_1 d^{4N}z=0.$$

Thus, taking the limits $\alpha\to 0$, $\alpha\to\infty$ and taking into account that

$$\delta(\xi^i_H)=\frac{1}{\pi^{2N}}\lim_{\lambda\to\infty}\sqrt{\lambda^{2N}\det g_{ij}}\, e^{-\lambda\xi^i_H g_{ij}\xi^j_H},$$

we obtain the DH localization formula

$$Z_0=\int_M e^H\sqrt{\det\omega_{ij}}d^{2N}x=\lim_{\alpha\to\infty}\int_{\mathcal{M}} e^{H-F-\lambda(H_2-F_2)}d^{4N}z$$

$$=\int_M e^H\delta(\xi_H)\sqrt{\det\omega_{ij}}\sqrt{\det\frac{\partial\xi^i_H}{\partial x^j}}\,d^{2N}x.$$

Generalization of the presented constructions for the path integrals can be accomplished by lifting them to the loop space similarly to the procedure in Refs. 1–3.

Then $H\to\int A_i dx^i-H\,dt$ (where $dA=\omega$), $\xi^i_H\to\xi^i_S=(\dot{x}^i-\xi^i_H)$, and the path integral localizes in the ordinary integral over the classical phase space.

Note that the representation of the initial integral in the form (2) formally coincides with the form of the integral from differential forms in the case where M is the *supermanifold*.[6] Note also that the present description is symmetric with respect to the initial and auxiliary coordinates. It can therefore be generalized to the super-Hamiltonian systems.

4. If on M both symplectic and Riemannian structures are defined, then on $\mathcal{M}$ one can also construct *even symplectic structures*:

$$\Omega_\alpha=\tfrac{1}{2}(\omega_{(\alpha)ij}+R_{ijkl}\theta^k\theta^l)dx^i\wedge dx^j+g_{ij}D\theta^i\wedge D\theta^j,\quad \alpha=0.2,$$

where $D\theta^i=d\theta^i+\Gamma^i_{kl}\theta^k dx^l$; R_{ijkl} and Γ^i_{kl} are correspondingly the curvature and the connection associated with the metric g_{ij} on M, and $\omega_{(0)ijk}\equiv\omega_{ij}$.

It is easy to see that $(H_0+F_2,\Omega_0,\mathcal{M})$, $(H_2+F_2,\Omega_2,\mathcal{M})$, and $(Q,\Omega^1,\mathcal{M})$ define the same supersymmetric dynamics (9), if q_{ij} is Lie-derived with ξ^i_H.

The example of supersymmetric dynamics with even and odd Hamiltonian structures (the one-dimensional Witten dynamics) was first proposed by Volkov *et al.*[7] Such dynamics were also considered in Ref. 8.

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