Charge distribution in quantum shot noise

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The charge which passes through a low-temperature quantum conductor over a fixed time has a binomial distribution. Fluctuations in the electron flux from the source and fluctuations due to scattering of electrons in the conductor are taken into account separately in the derivation. A generalization is made to the case of an arbitrary number of conduction channels and an arbitrary temperature. A generalized binomial (Bernoulli) distribution arises for the multichannel situation at absolute zero. In this distribution, the probabilities of the various outcomes are expressed in terms of multiparticle scattering probabilities.

Introduction. Under quantum transport conditions, the shot noise due to the discrete nature of charge is weaker than its classical value. The reason is the Pauli principle. Conforming to this principle, the electrons in a quantum conductor follow each other more regularly than they would in a classical case. This effect should be seen in its simplest form at a point contact,[1] in which the scattering is purely elastic, and the voltage satisfies $V \gg T/e$, where T is the temperature in the banks. Another interesting case is the scanning tunneling microscope.[2] In each case the system can be thought of as an elastic scatterer in a channel connecting two reservoirs of Fermi particles.[3] If the Coulomb interaction is ignored, the effect can easily be determined.[1,2,4] For the mean square fluctuation of the charge which passes over a time t, $q(t)$, we find

$$\langle\langle q^2(t)\rangle\rangle = \langle q^2(t)\rangle - \langle q(t)\rangle^2 = e(1-D)It, \tag{1}$$

where I is the average current, and $D=|A|^2$ is the transparency. Here A is the transmission amplitude. Expression (1) differs by a factor of $1-D$ from the known result for classical shot noise.

To see the features of quantum noise it is interesting to find the distribution of probabilities for various values of the charge and to compare this distribution with the Poisson distribution which describes classical noise. The problem has been studied by introducing an auxiliary spin in a system; this spin precesses in the magnetic field of the current and thereby measures the charge which has passed.[5] The probabilities found in the limit $T \to 0$ are nearly the same as a binomial distribution with the probabilities for two outcomes, $p=D$, $q=1-D$, and with a number of attempts $geVt/h$, where t is the observation time, and g the spin degeneracy. More precisely, we write the probability (P_m)for the passage of a charge me over a time t as follows:

$$P_m = \sum_N \rho(N) p^m q^{N-m} C_N^m. \tag{2}$$

In other words, we are writing it as binomial distributions with various numbers of attempts N, weighted with some distribution ρ (N). (Any distribution with positive values of m can formally be represented in this way.) It has been asserted[5] that at $T=0$ the distribution $\rho(N)$ is very narrow:

$$\text{(i)} \quad \langle N \rangle = geVt/h, \quad \text{(ii)} \quad \langle\langle N^2 \rangle\rangle = \frac{g}{2\pi^2} \ln E_F t/\hbar. \tag{3}$$

This assertion means that we can use the approximation $P_m = p^m q^{\langle N \rangle - m} C_{\langle N \rangle}^m$. The method of Ref. 5, used in deriving (2) and (3), is quite formal, so we would like to have a different derivation—one which would perhaps be less rigorous but which would clarify the meaning of the results. Finding this other derivation is our first goal here.

Fluctuations in the number of attempts. The relationship between the probabilities p and q, on the one hand, and the transparency D, on the other, follows from the problem of the scattering of a single particle, and it requires no explanation. It is interesting to look at expression (3.i) for the mean number of attempts. If we ignore the logarithmic fluctuations, (3.ii), the dependence of $\rho(N)$ on the observation time t is as it would be if the attempts were made periodically at a frequency geV/h. This picture can be only approximately correct, of course, since N actually fluctuates in a random manner. Let us evaluate the intensity of the fluctuations in N; we will show that at sufficiently low temperatures these fluctuations can be ignored in comparison with fluctuations stemming from scattering.

We first consider fluctuations in the equilibrium case, in which the potential difference across the banks is $V=0$. The fluctuations in the charge which passes over a time t, $q(t) = \int_0^t I(t')dt'$, are given by

$$\langle\langle q^2(t) \rangle\rangle = \int_{-\infty}^{\infty} \frac{4}{\omega^2} \sin^2 \frac{\omega t}{2} \langle\langle I_\omega^2 \rangle\rangle \frac{d\omega}{2\pi}, \tag{4}$$

where the fluctuations in the current are expressed in terms of the conductance G_ω in accordance with the fluctuation dissipation theorem: $\langle\langle I_\omega^2 \rangle\rangle = \hbar\omega G_\omega \coth \hbar\omega / 2T$. Substituting into (4), we find

$$\langle\langle q^2(t) \rangle\rangle = \begin{cases} \dfrac{2}{\pi} \hbar G_0 \ln \omega_0 t, & T \ll \hbar/t, \\ 2TG_0 t + \dfrac{2}{\pi} \hbar G_0 \ln \hbar\omega_0/T, & T \gg \hbar/t, \end{cases} \tag{5}$$

where ω_0 is a typical frequency of the dispersion of the conductance.

We can now find the fluctuations in the numbers of particles (N_L and N_R) which leave the reservoirs on the left and the right over a time t. For this purpose we consider a completely open channel, for which we have[3] $G_0 = ge^2/h$. In this case there are no fluctuations due to scattering, and expression (5) gives us the fluctuations in the equilibrium flux of particles which pass through the system without reflection. We can link the fluctuations in the charge $q(t)$ with fluctuations in the fluxes from the left and right banks, since we have $q(t)=e(N_L-N_R)$. At equilibrium, the average current is zero, i.e., $\langle N_L\rangle=\langle N_R\rangle$, and the two fluxes are statistically independent. We can thus write

$$\langle\langle N_L^2\rangle\rangle=\langle\langle N_R^2\rangle\rangle=\frac{1}{2e^2}\langle\langle q^2(t)\rangle\rangle.$$

We turn now to the nonequilibrum case, $V\neq 0$. We assume that the fluctuations in the fluxes N_L and N_R are the same as at equilibrium. This assumption is valid if the energy relaxation time in the banks is short enough that the particles which have passed have managed to become thermalized by the time they return to the contact. The difference between the fluxes, N_L-N_R, is known[3] to be $geVt/h$. Let us assume that the temperature T is very low. We then see from (5) that the condition $\langle\langle N^2_{L(R)}\rangle\rangle\ll|\langle N_L-N_R\rangle|$ holds for all sufficiently long times t; i.e., the fluctuations in the flux are extremely small. The physical reason for the uniformity of the flux is the Pauli principle, which forbids particles from passing through the contact simultaneously.

We now incorporate scattering again. The particles either pass through or are reflected, with probabilities $p=D$ and $q=1-D$. Expression (2), in which $\rho(N)$ is the distribution of the quantity N_L-N_R, is valid for the probability for the passage of several particles. At $T=0$ we find from (5) a logarithmic dependence of $\langle\langle N^2\rangle\rangle$ on t which is the same as (3). For the fluctuations in the number of particles distributed in accordance with (2) we easily find the expression

$$\langle\langle m^2\rangle\rangle=p^2\langle\langle N^2\rangle\rangle_\rho+pq\langle N\rangle_\rho. \tag{6}$$

The first term here stems from fluctuations in the flux of incident particles, and the second from scattering. We see that the nonuniformity of the flux can be ignored if $p\langle\langle N^2\rangle\rangle\ll q\langle N\rangle$, i.e., if

$$DT\ll(1-D)eV. \tag{7}$$

This result is the condition under which the scatter in the number of attempts is less important than the scatter due to the binomial distribution with the mean number of attempts. Under condition (7), P_m is given by a binomial distribution $p^mq^{\langle N\rangle-m}C^m_{\langle N\rangle}$ with the mean number of attempts.

This derivation of the binomial distribution reveals that the only important factors here are the elastic nature of the scattering and the low temperature. The logarithmic fluctuations in the number of attempts at $T=0$ are totally unrelated to a one-dimensional nature of the situation, as has previously been assumed.[5] They are found directly from the Callen–Welton formula, in identical ways for an arbitrary dimensionality of the space and for an arbitrary geometry of the conductor. These fluctuations are nevertheless related to fluctuations in the density of one-dimensional fermions, but along the time axis, not the spatial axis.

The binomial distribution becomes a Poisson distribution in two limiting cases: $D\to 0$ and $D\to 1$. The first of these cases corresponds to classical shot noise, and the second to transport in a system with almost no reflection. (In the second case, the Poisson distribution is the distribution of reflected particles, not that of transmitted particles.)

General expression for the multichannel case. We can derive an analytic expression for the charge distribution which is valid for an arbitrary relation between T and eV and also for an arbitrary number of channels. We assume that we have M current leads (we will speak of them as channels) and a scattering matrix A_{jk} (which is an $M\times M$ unitary matrix). The distribution of particles leaving the channels is a Fermi distribution, $n_j(E)=1/(e^{(E-\mu_j)/T}+1)$, where μ_j is the chemical potential in the jth reservoir. We introduce M real variables λ_j, $j=1,\ldots, M$, and we construct a characteristic function whose Fourier transform gives us the probabilities for various values of the charge. We define the matrices $\widetilde{A}$ and $\mathbf{n}_E$ by $\widetilde{A}_{jk}=e^{i(\lambda_j-\lambda_k)}A_{jk}$ and $(\mathbf{n}_E)_{jk}=n_j(E)\delta_{jk}$. We consider the determinant

$$\chi_E(\boldsymbol{\lambda})=\det(\mathbf{1}-\mathbf{n}_E+\mathbf{n}_E A^+\widetilde{A}), \tag{8}$$

which, as we see, represents a characteristic distribution function of the transmitted charge for particles with energies in an infinitely narrow band near E. We can show that the complete characteristic function is

$$\chi(\boldsymbol{\lambda})=\exp\left[gt\int \ln \chi_E(\boldsymbol{\lambda})\frac{dE}{2\pi\hbar}\right]. \tag{9}$$

To do this, we expand determinant (8):

$$\chi_E(\boldsymbol{\lambda})=\sum_{i_1,\ldots,i_k}(A^+\widetilde{A})^{i_1,\ldots,i_k}_{i_1,\ldots,i_k}\prod_{i\neq i_\alpha}1-n_i(E)\prod_{i=i_\alpha}n_i(E). \tag{10}$$

The summation is over all sets of i_α which are not equal to each other: $1\leqslant i_\alpha\leqslant M$, $i_\alpha\neq i_\beta$ with $\alpha\neq\beta$ (α, $\beta=1,\ldots,$ $k\leqslant M$). The symbols

$$S^{\,j_1,\ldots,j_k}_{\,i_1,\ldots,i_k}$$

represent the minor of the matrix S with rows $i_1,\ldots,i_k$ and columns $j_1,\ldots,j_k$. We now note that we have

$$(A^+\widetilde{A})^{i_1,\ldots,i_k}_{i_1,\ldots,i_k}=\sum_{j_1,\ldots,j_k}e^{i(\lambda_{j_1}+\ldots+\lambda_{j_k}-\lambda_{i_1}-\ldots-\lambda_{i_k})}|A^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}|^2. \tag{11}$$

The summation over $j_1,\ldots,j_k$ is the same as in (10). We substitute (11) into (10):

$$\chi_E(\boldsymbol{\lambda})=-\sum_{i_1,\ldots,i_k,j_1,\ldots,j_k}e^{i(\lambda_{j_1}+\ldots+\lambda_{j_k}-\lambda_{i_1}-\ldots-\lambda_{i_k})}P_{i_1,\ldots,i_k|j_1,\ldots,j_k}, \tag{12}$$

where

$$P_{i_1,\ldots,i_k|j_1,\ldots,j_k}=|A^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}|^2\prod_{i\neq i_\alpha}1-n_i(E)\prod_{i=i_\alpha}n_i(E)$$

contains the square amplitude for the transition of k particles from channels $i_1,...,i_k$ into channels $j_1,...,j_k$ and the product of the probabilities that the particles arrive from channels $i_1,...,i_k$, while nothing arrives from other channels. Accordingly, the quantity $P_{i_1,...,i_k|j_1,...,j_k}$ is simply the probability that k charges go from channels $i_1,...,i_k$ to $j_1,...,j_k$. The probabilistic meaning of Eq. (8) has been clarified. To go from (8) to (9) we note that states with different energies are scattered independently, without interference. The flux of particles with energies in the interval δE is given by $\delta N=(gt/2\pi\hbar)\delta E$. In light of the comment which we just made, we thus find (9).

Probabilities $P_{i_1,...,i_k|j_1,...,j_k}$ which are the same as ours have been studied elsewhere[6] in the context of the problem of fluctuations in the charge which has passed in the one-dimensional case ($M=2$). Those probabilities were used to calculate the intensity of the shot noise in (1). However, the characteristic function and the charge distribution were not discussed.

Example. We consider a three-terminal system[6] in the form of the letter Y, which contains three channels. We seek the characteristic function. For simplicity we assume $T=0$, and we number the channels in such a way that the conditions $\mu_1>\mu_2>\mu_3$ hold. The scattering matrix A_{jk} is a 3×3 unitary matrix. Since $T=0$, the products of $n_j(E)$ and $1-n_j(E)$ in $P_{i_1,...,i_k|j_1,...,j_k}$ are equal to 1 or 0, depending on the relation between E and μ_1, μ_2, μ_3. We therefore have

$$\chi_E(\lambda)=\begin{cases}1, & E>\mu_1 \text{ or } \mu_3>E\\ \chi_1(\lambda)=P_{1|1}+P_{1|2}\,e^{i(\lambda_2-\lambda_1)}+P_{1|3}\,e^{i(\lambda_3-\lambda_1)}, & \mu_1>E>\mu_2\\ \chi_2(\lambda)=P_{12|12}+P_{12|13}\,e^{i(\lambda_3-\lambda_2)}+P_{12|23}\,e^{i(\lambda_3-\lambda_1)}, & \mu_2>E>\mu_3\end{cases} \tag{13}$$

According to (9) and (13), the characteristic function can be factorized:

$$\chi(\lambda)=\chi_1(\lambda)^{g(\mu_1-\mu_2)t/h}\chi_2(\lambda)^{g(\mu_2-\mu_3)t/h}. \tag{14}$$

This result means that the distribution arises as a consequence of two independent random processes, each constituting a Bernoulli process with three outcomes. The probabilities for these outcomes are $P_{1|i}=|A_{1i}|^2$ and $P_{12|ij}=|A_{1i}A_{2j}-A_{1j}A_{2i}|^2$. The frequency of trials is $g(\mu_1-\mu_2)/h$ for the first process and $g(\mu_2-\mu_3)/h$ for the second.

Interestingly, in the case of three channels the two-particle probabilities can be expressed in terms of one-particle probabilities. This circumstance is a consequence of the unitary nature of the scattering matrix and of some known identities for minors, such as $A_{12}A_{23}-A_{13}A_{22}=(A^{-1})_{13}\det(A)$. We thus have $P_{12|12}=P_{3|3}$, $P_{12|13}=P_{3|2}$, and $P_{12|23}=P_{3|1}$. This result can be understood physically by switching from electrons to holes. Two filled channels and one empty one are transformed into one filled channel (filled with holes) and two empty ones, and the one-particle scattering amplitudes for the holes are the complex conjugates of the electron scattering amplitudes.

We can treat the case of an arbitrary number of channels, M, at $T=0$ in a corresponding way. In this case we have $M-1$ independent Bernoulli processes, corresponding to different numbers of filled channels. The difference between the example we just looked at is that at $M>3$ the multiparticle probabilities generally cannot be expressed in terms of one-particle probabilities. (But they can, of course, be expressed in terms of one-particle amplitudes.)

We conclude with a few words about our previous paper,[7] in which a distribution was calculated by a method of projection onto the eigenfunctions of an operator representing the charge which has passed in a fixed time, $\hat{Q}_t=\int_0^t \hat{j}(x, t')dt'$. This situation corresponds to the standard approach to the description of measurements in quantum mechanics. That calculation leads to a binomial distribution with a fractional charge quantum and with outcome probabilities different from D and $1-D$, but it does give the first and second moments correctly. The reason for the discrepancy with the correct result, found through a rigorous calculation[5] and confirmed in the present study, is that the projection onto eigenstates of the operator $\hat{Q}_t$ actually does not correspond to any real scheme for measuring charge; i.e., it is nonphysical. The question of charge measurements in this case is a rather subtle one and requires an analysis involving the incorporation of a measuring instrument in the Hamiltonian of the problem of Ref. 5. At the same time, we note that although projection onto the eigenvalues of $\hat{Q}_t$ cannot be regarded as a correct approach, it nevertheless leads to a binomial distribution. In other words, it correctly describes the main aspect of the problem: the small fluctuations in the number of attempts.

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