Quantum birth of the universe and initial conditions for inflation

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Paths describing evolution of the universe in the classically forbidden region in the (a,φ) plane have been found. A tunneling occurs in such a way that the scalar-field potential $v(\varphi)$ increases "on the average" along the tunneling path. This circumstance satisfies the initial conditions required for inflation of the classical universe. The probabilities for various tunneling paths of the universe are compared.

The goal of this letter is to find paths in the (a, φ) plane (a is a scaling factor, and φ is a uniform scalar field) along which the universe can tunnel in the classically forbidden region $a<(2l^2v)^{-1/2}$, where $l^2=4\pi G/3$, G is Newton's constant, and $v(\varphi)$ is the potential of the field φ. Tunneling in the a direction is usually interpreted as a quantum birth of the universe.[1-3] A wave function $\Psi(a, \varphi)$ which is localized near the maximum of the potential $v(\varphi)$ and which describes an inflation of a classical universe[6,7,8] was found in Refs. 4 and 5. In this letter we offer some other solutions, which also lead to an inflation in the classically allowed region. It can be shown[9] that the wave function in the classically forbidden region is localized along the most probable tunneling paths; we wish to determine these paths.

The Lagrangian in the minisuperspace model, which depends on the two variables a and φ, is

$$L=(-g)^{1/2}\left\{\frac{1}{2}\dot{\varphi}\dot{\varphi}-v(\varphi)-\frac{1}{16\pi G}R\right\} \tag{1}$$

(R is the curvature scalar). The superior dot means differentiation with respect to physical time. Lagrangian (1) leads to the Wheeler–de Witt equation[10]

$$\left\{-l^2\frac{\partial^2}{\partial a^2}+a^{-2}\frac{\partial^2}{\partial\phi^2}-2va^4+l^{-2}a^2+\alpha a^{-2}\right\}\Psi(a,\varphi)=0, \tag{2}$$

where the constant α incorporates possible quantum corrections.[11] Unless otherwise stipulated below, we are assuming $\alpha=0$.

In the forbidden region the wave function is concentrated along a one-parameter family of field configurations $\varphi[\lambda(t)]$, $a[\lambda(t)]$. The introduction of a parameter λ can reduce the dynamics of the fields to the dynamics of a system with one degree of freedom,[12,13] $\lambda(t)$. The effective Lagrangian for $\lambda(t)$ is found by substituting $\varphi=\varphi[\lambda(t)]$, $a=a[\lambda(t)]$ in Lagrangian (1):

$$L_{\rm eff}=\frac{m(\lambda)\dot{\lambda}\dot{\lambda}}{2}-V(\lambda),$$

$$m(\lambda)=a^3\left[\frac{\partial\varphi}{\partial\lambda}\frac{\partial\varphi}{\partial\lambda}-l^{-2}a^{-2}\left\{\frac{\partial a}{\partial\lambda}\right\}^2\right], \tag{3}$$

$$V(\lambda)=a^3\left[v(\varphi)-\frac{3}{8\pi G}a^{-2}\right].$$

Lagrangian (3) looks exactly like the quantum-mechanical Lagrangian with one degree of freedom. The Hamiltonian for λ is

$$\hat{H}(\lambda)=-\frac{1}{2m(\lambda)}\frac{\partial^2}{\partial\lambda^2}+V(\lambda). \tag{4}$$

The "mass" $m(\lambda)$ is not positive definite. If $m(\lambda)>0$, the classically forbidden (allowed) region is determined by the condition $V(\lambda)>0(<0)$. If $m(\lambda)<0$, then the classically forbidden (allowed) region is at $V(\lambda)<0(>0)$. There is an important point here: The sign of $m(\lambda)$ depends not only on the point in the (a,φ) plane but also on the direction $\partial a/\partial\varphi$. By analogy with 1D quantum mechanics we find the tunneling probability P in the semiclassical approximation:

$$P\simeq\exp(-S),\quad S=\int_{\lambda_1}^{\lambda_2}d\lambda[2mv]^{1/2}. \tag{5}$$

We then seek an extremum (a minimum) of S in order to determine the most probable tunneling paths. The condition $\delta S=0$ leads to Einstein's equations written in the imaginary time τ, which is given in terms of λ: $\partial\tau/\partial\lambda=[m/2V]^{1/2}$:

$$\frac{\partial\varphi^2}{\partial\tau^2}+3a^{-1}\frac{\partial a}{\partial\tau}\frac{\partial\varphi}{\partial\tau}=\frac{\partial v}{\partial\varphi},$$
$$-a^{-2}\left[\frac{\partial a}{\partial\tau}\right]^2+a^{-2}=\frac{8\pi G}{3}\left\{-\frac{1}{2}\left[\frac{\partial\varphi}{\partial\tau}\right]^2+v(\varphi)\right\}. \tag{6}$$

It is easy to see that Einstein's equations in imaginary time can be solved only if $\text{sign}(m)=\text{sign}(V)$; this condition is the same as the condition for the classically forbidden region, given above. Einstein's equations in real time, on the other hand, can be solved only in the classically allowed region. The boundary between these regions is defined by the equation $V(\lambda)=0$.

We take the potential $v(\varphi)$ to be $v(\varphi)=v_0+2^{-1}\mu^2\varphi^2$. At small values of a we find $a\approx\tau$ from the second equation in (6). In this approximation the two linearly independent solutions are

$$\varphi_1=\tau^{-1}J_1(|\mu|\tau),\quad \varphi_2=\tau^{-1}N_1(|\mu|\tau),\quad \mu^2<0, \tag{7}$$

$$\varphi_1=\tau^{-1}I_1(\mu\tau),\quad \varphi_2=\tau^{-1}K_1(\mu\tau),\quad \mu^2>0. \tag{8}$$

Here J and N are the Bessel and Neumann functions, and I and K are modified Bessel functions. We first consider the case $\mu^2>0$, i.e., the case in which the potential has a minimum at $\varphi=0$. The solution $\varphi_1=\tau^{-1}I_1(\mu\tau)$ describes a curve in the (a, φ) plane which starts from some point $(a=0, \varphi_{in})$ and then runs off to large values of $|\varphi|$. The solution $\varphi_2=\tau^{-1}K_1(\mu\tau)$ is based on the assumption that the initial values of the field φ are infinite and must therefore be excluded. This solution may be meaningful, however, if we have $\alpha>0$ in Wheeler-de Witt equation (2).

We now consider a potential $v(\varphi)$ which has a maximum at $\varphi=0$. In this case only the solution $\varphi_1=\tau^{-1}J_1(|\mu|\tau)$ satisfies the condition that φ be finite at $a=0$. This solution describes a curve in the (a, φ) plane which begins at some $v<v_{max}$ and then approaches the crest of the potential, where it begins to oscillate with a damped amplitude. The solutions φ_1 from (7), (8) satisfy the boundary condition $\partial\varphi/\partial a|_{a=0}=0$. The solutions written above are valid until the approximation $a\approx\tau$ breaks down.

We can now determine paths which deviate slightly from a straight path. A straight path is given by the following solution of system (6):

$$\varphi=0, \quad a(\tau)=\frac{\sin[(2l^2v)^{1/2}\tau]}{(2l^2v)^{1/2}}, \tag{9}$$

where $v=v(0)$. We assume $(1/2)[\partial\varphi/\partial\tau]^2\ll v(0)$. Using the expression in (9) for $a(\tau)$, we then find a solution for φ which is finite at $a=0$:

$$\varphi=C\frac{d}{dx}P_\nu(x), \quad \nu=-\frac{1}{2}-\left[\frac{9}{4}-\frac{\mu^2}{2l^2v}\right]^{1/2}, \quad x=\cos[(2l^2v)^{1/2}\tau], \tag{10}$$

where C is an arbitrary constant. This solution is approximate over the entire region $0<x<1$ for small values of C.

Let us estimate the probabilities for the various paths. From expression (5) we find the probability for straight path (9):

$$P_0\approx\exp(-S_0), \quad S_0=2/3l^4v(0). \tag{11}$$

It follows from (5) that, in the case of a potential maximum ($\mu^2<0$), curved path (7) or (10) is less probable than a straight path. For a slightly curved path we have

$$S=S_0-\mu^2\int a^3\varphi^2 d\tau. \tag{12}$$

If the universe is born with $v=v_{max}$, then the most probable evolution in the classically forbidden region $a<(2l^2v)^{-1/2}$ is thus an evolution along straight path (10). A universe born with $v=v_{max}$ can begin to evolve in the classically forbidden region with $\varphi=\varphi_{in}$, $v(\varphi_{in})<v_{max}$ and then approach $v=v_{max}$ along the trajectory $\varphi_1=\tau^{-1}J_1(|\mu|\tau)$. However, this path is less probable than the straight path.

Let us consider the birth of a universe with $v=v_{min}$, i.e., near a potential minimum, and let us assume that the forbidden region begins with $a_1>0$. This condition corresponds to $\alpha>0$ in Eq. (2). The solution of system (6) is a linear combination of $\varphi_1=\tau^{-1}I_1(\mu\tau)$ and $\varphi_2=\tau^{-1}K_1(\mu\tau)$. It follows from expression (5) for S that the most likely path is one which passes through the region of large values of potential $|v|$, so the evolution described by the path φ_2 is the most probable. We have thus shown that the most probable tunneling path is "on the average" directed toward large values of the potential of the scalar field, $v(\varphi)$. We would thus expect that after the tunneling the field φ would most likely be localized either at $v=v_{max}$ or at large values of $v(\varphi)$. Since the path φ_1 from (7) is the same as (9) in the final stage of the tunneling, it, too, leads to an inflation in the classical region. Note that the different tunneling paths lead to different excitations of nonuniform modes of the field φ, so they lead to different large-scale perturbations of the energy density after the end of the inflation. Calculations show that the creation of particles is insignificant for the straight path in (9) and for the slightly curved paths in (10), and nonuniform modes are in the vacuum state at the end of the tunneling. For the path φ_2 from (8), however, we would expect a substantial creation of particles. This circumstance may have implications for predictions regarding the spectrum of irregularities in a model of chaotic inflation.

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