Multiplier distribution for a turbulence dissipation field

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A model probability distribution is derived for the multiplier of the field of the energy dissipation rate in very turbulent flows. This distribution, which depends on a single adjustable parameter, is compared with experiment.

The small-scale structure of well-developed hydrodynamic turbulence has been the subject of numerous theoretical and experimental studies (see the reviews[1,2]). As Chhabra and Sreenivasan[3] have pointed out, none of the existing theoretical models can explain the experimental distributions of the multiplier of the dissipation field from the inertial interval which were found in that study.

We show below that the results of an invariant probabilistic modeling[2] of the rate of energy dissipation in very turbulent flows of an incompressible liquid can be reconciled satisfactorily with experiment.[3]

The model equation

$$\Phi''+2(\lambda_1+\lambda_{2q})\Phi'+\lambda_{3q}(q-1)\Phi=0 \qquad (1)$$

was derived in Ref. 2 for the function $\Phi(q, x)$ of moments of the normalized dissipation. Here $\Phi(q, x)=\langle\epsilon^q\rangle/\langle\epsilon\rangle^q$; the angle brackets mean a probabilistic average; ϵ is the rate at which kinetic energy is dissipated; the prime means differentiation with respect to x; x is the logarithm of the turbulent Reynolds number $\mathrm{Re}=k^2/\langle\epsilon\rangle\nu$; k is the average energy of the turbulence; and ν is the molecular viscosity coefficient. The coefficients λ_1, λ_2, and λ_3 generally depend on the Reynolds number.

The conclusion[2] that there cannot exist a finite, nonzero asymptotic value of the parameters $\lambda_{1,2,3}$ in the limit $x\to\infty$ was based on the implicit assumption that the dissipation probability distribution has statistical moments of arbitrary orders (both positive and negative). In addition, the multiplier distribution in the scaling interval cannot be independent of the turbulent Reynolds number of the flow. It is shown below that, if dissipation moments of certain orders do not exist (if they diverge), then the parameters of the multiplier probability distribution is not necessarily a function of the Reynolds number in the inertial interval, and the distributions themselves agree fairly well with experiment.

Following Ref. 4, we consider the multiplier of the field of the rate of turbulence energy dissipation, $e_{r,m}=\langle\epsilon\rangle_r/\langle\epsilon\rangle_m$, under the condition $r<m$, where $\langle\epsilon\rangle_r$ means the dissipation averaged over a spatial region with a length scale r. If we assume that in some scaling interval (inertial interval) the distribution of the coefficient $e_{r,m}$ depends on only the ratio r/m, and if the succeeding multiplications are independent, then[4]

$$\langle (e_{r,m})^q \rangle = (m/r)^{\mu_q} \quad \text{for } \eta \ll r,m \ll L. \tag{2}$$

Here $\eta = \langle \epsilon \rangle^{-1/4} \nu^{3/4}$ and $L = k^{3/2}/\langle \epsilon \rangle$ are respectively the internal Kolmogorov scale and the external scale of the turbulence.[5] The parameter μ_q is a universal function of q, independent of the Reynolds number.

A quantity analogous to the coefficient $e_{r,m}$ was used in Ref. 3: $M(b) = e_{r,m}/b$, where $b = m/r$. From the condition that ϵ be nonnegative one finds[4] $0 \leqslant M(b) \leqslant 1$.

The minimum size of the vortices in a turbulent flow is $\sim \eta$, and the average of the field ϵ over a length scale $\sim L$ can be assumed to be the same as $\langle \epsilon \rangle$. From Eq. (2) one finds[2]

$$\Phi(q,x) \sim (L/\eta)^{\mu_q}.$$

The ratio of the external scale to the Kolmogorov scale is equal to the turbulent Reynolds number raised to a power of 3/4. We can then write

$$\Phi(q,x) = C(q) \exp\{3/4 \mu_q x\}. \tag{3}$$

If the parameter μ_q does not depend on the Reynolds number, then substitution of expression (3) into Eq. (1) yields

$$\mu_q = \frac{4}{3}[-(\lambda_1 + \lambda_{2q}) \pm ((\lambda_1 + \lambda_{2q})^2 - \lambda_{3q}(q-1))^{1/2}], \tag{4}$$

where the parameters $\lambda_{1,2,3}$ are constants.

We denote by $P(M,b)$ the distribution of the multiplier M. From Eq. (2) and from the limitation on the range of M values we find

$$\int_0^1 P(M,b) M^q dM = \exp\{\ln(b)(\mu_q - q)\}. \tag{5}$$

Since the coefficient M has an upperbound, the distribution $P(M,b)$ has statistical moments of all positive orders. From Eq. (4) we then find $\lambda_2^2 - \lambda_3 \geqslant 0$.

Let us consider the probability distribution $G(Y,b)$ of the quantity $Y = -\ln(M)$. The functions P and G are related by $P(M,b) = G(-\ln(M),b)/M$. From Eq. (5) we find the following equation for the function $G(Y,b)$:

$$\int_0^\infty G(Y,b) e^{-qY} dY = \exp\{\ln(b)(\mu_q - q)\}. \tag{6}$$

We assume $\lambda_2^2 - \lambda_3 > 0$; then in the limit $q \to \infty$ we have

$$\int_0^\infty G(Y,b) e^{-qY} dY \sim \exp\left\{-q \ln(b)\left[1 + \frac{4}{3}\lambda_2 \pm (\lambda_2^2 - \lambda_3)^{1/2}\right]\right\} \times \left(1 + \sum_{k=1}^{\infty} A_k q^{-k}\right),$$

and from the properties of the Laplace transform[6] we find that the function $G(Y)$ and therefore the function $P(M)$ must have a singularity in the form of a Dirac δ-function.

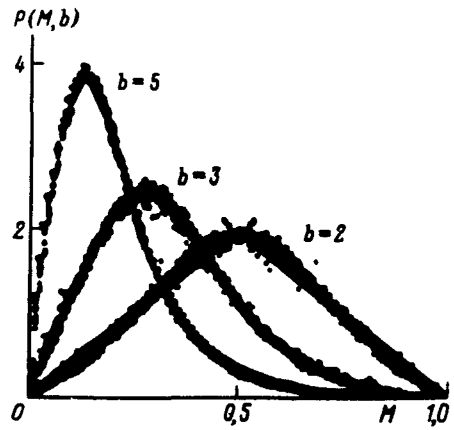

FIG. 1. Distribution of the multiplier M for various values of the multiplier parameter b according to the data of Ref. 3. The solid curve is a triangular distribution which was proposed in Ref. 3 as an approximation of the distribution in the $b=2$ case.

Figure 1 shows the results of an experimental determination of the M distribution for various values of the parameter b. We see that the $P(M,b)$ distributions are unimodal and do not have singularities (which would be manifested experimentally as a second sharp peak). We therefore conclude

$$\lambda_2^2-\lambda_3=0. \tag{7}$$

From the hypothesis of the independence of multiplications[4] we find that the distribution G is infinitely divisible,[7] i.e., it is the distribution of a quantity which can be written as the sum of an arbitrary number of independent, identically distributed terms. A necessary and sufficient condition for the function $\varphi(q)$ to be the Laplace transform of an infinitely divisible distribution[7] is the condition that we can write the following:

$$\varphi(q)=\exp\{-\Psi(q)\},$$

where $\Psi(0)=0$, and the function $\frac{d}{dq}\Psi(q)$ is the Laplace transform of some nonnegative measure. We then find from Eqs. (4), (6), and (7) that of the two solutions in Eq. (4) we need to choose that which corresponds to the minus sign in front of the radical.

The experimental results of Ref. 3, shown in Fig. 1, are evidence that the function $P(M,b)$ does not vanish identically for some $0<M_0<1$. In this case the Laplace transform[6] of the function G does not necessarily contain a factor which is exponential in q. Using (4), we find $\lambda_2=-3/4$, and from (7) we find $\lambda_3=9/16$.

Finally, for the function μ_q we find

$$\mu_q=\frac{4}{3}\left[-\lambda_1+\frac{3}{4}q-\left(\lambda_1^2+q\left(\frac{9}{16}-\frac{3}{2}\lambda_1\right)\right)^{1/2}\right]. \tag{8}$$

From the obvious normalization condition $\mu_0=0$ we find $\lambda_1\leqslant 0$. Since the quantity μ_q is real, from (8) we find

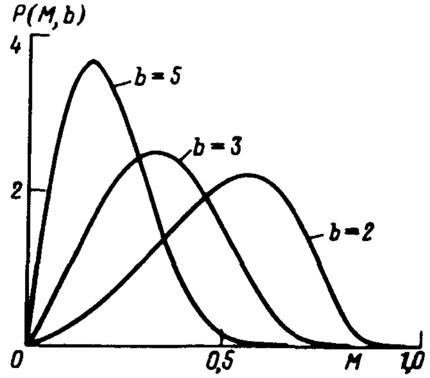

FIG. 2. Distribution of the multiplier M according to Eq. (9) with $\lambda_1=-3.5$.

$$q \geqslant -\lambda_1^2 \Big/ \left(\frac{9}{16}-\frac{3}{2}\lambda_1\right).$$

The coefficient λ_1 can be related to the intermittency parameter μ. If σ_r^2 is the variance of the logarithm $\langle\epsilon\rangle_r$, we find[1,2] from (2)

$$\sigma_r^2=\mu\ln(L/r), \quad \mu=\left(\frac{d^2}{dq^2}\mu_q\right)\Big|_{q=0},$$

and from Eq. (8) we find

$$\mu=\frac{3}{4}\left(\frac{3}{8}-\lambda_1\right)^2 \Big/ (-\lambda_1)^3.$$

Substituting (8) into (6), we find the infinitely divisible distribution $G(Y,b)$ (see problem 5 in Chapter XIII in Ref. 7). Switching to the variable M, we find the multiplier distribution $P(M,b)$:

$$P(M,b)=(2\pi)^{-1/2}\rho a Y^{-3/2}M^{-1}\exp\left\{-\frac{1}{2}[\rho a Y^{-1/2}+\lambda_1 a^{-1}Y^{1/2}]^2\right\}; \qquad (9)$$

here $\rho=\frac{4}{3}\ln(b)$, $a=(\frac{9}{32}-\frac{3}{4}\lambda_1)^{1/2}$, and $Y=-\ln(\mathrm{M})$.

Figure 2 shows distributions $P(M,b)$ from (9) for the values $b=2$, 3, and 5. We selected the value $\lambda_1=-3.5$; in this case we have $\mu\approx0.26$. Figure 3 shows distributions of the multiplier M for $b=2$ and for the parameter values $\lambda_1=0$, -2, and -4. Comparison of the theoretical curves (Fig. 2) with the experimental curves (Fig. 1) reveals a satisfactory agreement.

Finally, we wish to stress that such theoretical models of the intermittency of a small-scale turbulence as the log-normal model,[8] the β-model,[9] the random β-model,[10] and the p-model[1] are at odds with the experimental distribution of the dissipation multiplier.[3] The distribution found in the present letter leads to a satisfactory description of the distribution $P(M,b)$ of the dissipation field multiplier M for various values of b with only a single adjustable parameter. The intermittency parameter μ can be chosen as this adjustable parameter.

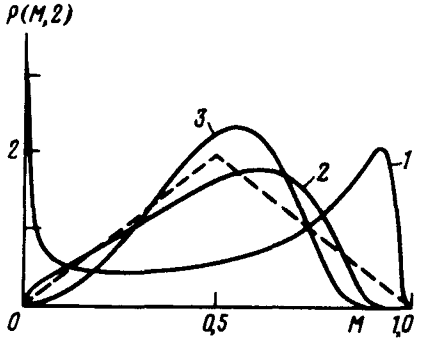

FIG. 3. Distribution of the multiplier M for $b=2$. Curves *1*, *2*, and *3* were found from Eq. (9) with $\lambda_1=0,-2,-4$. The dashed line shows a triangular distribution.

The model distributions vanish more rapidly than the measured ones in the limit $M\to 1$. It is not difficult to show that this circumstance leads to values of the higher-order structure functions of the velocity field which are higher than the experimental ones. This discrepancy between theory and experiment may be a consequence of a weak dependence of the multiplier distribution on the Reynolds number of the flow in the scaling interval; that possibility was not considered in the present letter.

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