Chiral field theory for describing fluctuating surfaces or membranes

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The relaxation dynamics of fluctuations in the shape of a membrane is formulated on the basis of a Langevin equation for a matrix chiral field constructed with the help of a local Frenet frame of reference (n-hedron). The concept of a chiral field proves useful, making it possible to construct a formally closed scheme for calculating arbitrary correlation functions for fluctuations in the shape of an arbitrary (not necessarily fixed) topology. Information on the internal geometry of the surface is used in the form of an explicit dependence of the correlation functions on chiral currents. In its general form the method is illustrated by a proof of the fluctuation dissipation theorem. A binary correlation function of fluctuations in the average curvature is found.

1. Membranes are entities of molecular thickness and otherwise macroscopic dimensions. In other words, a 2D surface in a 3D space can be used to represent a membrane mathematically. It is therefore natural to describe the properties of membranes (and, of course, those of various lyotropic systems formed from them[1]) by the methods of the classical theory of surfaces.

A fundamental structure used in surface theory[2] is the local n-hedron $\hat{X}$ formed by the vectors tangent to the surface (τ_1, τ_2) and the normal to the surface ($\mathbf{N}$):

$$\hat{X}=\{\partial\mathbf{r}/\partial\sigma^1,\ \partial\mathbf{r}/\partial\sigma^2,\ \mathbf{N}\}. \tag{1}$$

The radius vector $\mathbf{r}$ specifies an arbitrary point on the membrane, and $\sigma^\mu(\mu=1,2)$ are the coordinates on the surface.

The change in the local n-hedron from point to point on the surface is described by the Gauss–Weingarten equations[2]

$$\partial_\mu \hat{X}=\hat{j}_\mu \hat{X};\quad \mu=1,2,\quad \partial_\mu=\partial/\partial\sigma^\mu. \tag{2}$$

The matrices j_μ are determined by the geometry of the surface. They satisfy integrability conditions:

$$\partial_1 \hat{j}_2-\partial_2 \hat{j}_1+[\hat{j}_1,\hat{j}_2]=0 \tag{3}$$

where [.,.] is a commutator. The procedure for finding the explicit form of the matrices $\hat{j}_\mu$ is straightforward in principle, but quite laborious. In general, the components of these matrices are specified by Christoffel symbols and by the tensors of first and second quadratic forms of the surface.

Expressions (1) and (2) can be used to determine all the properties of a surface in the theory of the chiral field specified by the matrices $\hat{X}$, which belong to the GL(3, R) group. For example, the standard expression for currents in chiral field theory[3] follows from (2):

$$\hat{j}_\mu=\partial_\mu \hat{X}\hat{X}^{-1}. \tag{4}$$

2. It turns out that the analogy with a chiral field is not purely formal. It leads to several useful conclusions. Those questions will be the topic of a separate study; here we wish to call attention to several points of fundamental interest.

A fundamental role is played in the theory of membranes by the Helfrich Hamiltonian,[4] which describes the energy of a membrane as a function of its shape:

$$E_H=\int d^2x\left\{\frac{1}{2}\kappa(k_h)^2+\bar{\kappa}k_g\right\}. \tag{5}$$

Here κ and $\bar{\kappa}$ are elastic moduli of the membrane, k_h is the mean curvature, and k_g is the Gaussian curvature. [For simplicity we are assuming bilayer, i.e., symmetric, membranes, so there is no contribution linear in k_h in (5).] It is the ability to write Helfrich energy (5) in terms of chiral current $\hat{j}_\mu$ alone that makes the concept of a chiral field extremely useful for describing membranes. We write

$$k_h=\mathrm{Tr}(\hat{\varphi}^{(2)}\hat{g}^{-1}), \quad k_g=\det(\hat{\varphi}^{(2)}\hat{g}^{-1}),$$

where $\hat{\varphi}^{(2)}$ is the second quadratic form of the surface, and $\hat{g}$ is the metric tensor (the first quadratic form).

The integral of the Gaussian curvature is known to be a topological invariant. If we are dealing with a surface of fixed topology, Helfrich energy (5) can therefore be written

$$E_H=\int d^2x a^{\mu\nu}_{ijkl}\, j^\mu_{ij} j^\nu_{kl}=\frac{1}{2}\kappa\int d^2x(j^1_{31}+j^2_{32})^2. \tag{6}$$

This last expression is the definition of the matrix of coefficients $a^{\mu\nu}_{ijkl}$.

The equilibrium shape of the surface is governed by the condition for a minimum of functional (6); this condition is

$$\nabla_\mu(\hat{a}^{\mu\nu}\hat{j}_\mu)=0, \tag{7}$$

where ∇_μ is the covariant derivative

$$\nabla_\mu=\partial_\mu+[\,j_\mu,]. \tag{8}$$

3. To describe fluctuations of the membrane we define, in accordance with (2), a current component $\hat{j}_0$ which specifies the time evolution of the shape of the membrane:

$$\hat{j}_0=\partial_t\hat{X}\hat{X}^{-1}. \tag{9}$$

Denoting by the matrix $\delta\hat{v}$ the variation of the chiral field,

$$\hat{X}\rightarrow\hat{X}(1+\delta\hat{v}),$$

we can write

$$\hat{j}_{\mu}\rightarrow\hat{j}_{\mu}+\nabla_{\mu}\delta\hat{v} \quad \text{and} \quad \hat{j}_{0}=\partial_{t}\partial\hat{v}. \tag{10}$$

It was shown in Ref. 5 that for an isolated membrane at equilibrium with a solution there exists a single softest surface mode, which describes a pure relaxation of the surface to an equilibrium shape. Significantly, this remains the case in which nonlinear effects are taken into account. In the leading hydrodynamic approximation all the other, harder modes can thus be eliminated. As a result, we find an effective equation of Langevin dynamics:

$$\partial_{t}\delta\hat{v}=\gamma(\delta E_{H}/\delta\hat{v})+y. \tag{11}$$

Here γ is an effective kinetic coefficient, and y is a Gaussian random noise.

Using definitions (4) of the chiral currents and the explicit expression for the Helfrich energy, (6), we can put this equation in a closed form convenient for calculations:

$$\partial_{t}\delta\hat{v}=\nabla_{\mu}(\hat{a}^{\mu\nu\hat{j}}\hat{j}_{\mu})+y. \tag{12}$$

In the simplest case of small fluctuations $\delta\hat{v}$, the linearized right side of Eq. (12) is

$$\gamma\{\partial_{\mu}(\hat{a}^{\mu\nu}\nabla_{\nu}\delta\hat{v})+[\nabla_{\mu}\delta\hat{v},\hat{a}^{\mu\nu}\hat{j}_{\nu}^{(0)}]+[\hat{j}_{\mu}^{(0)},\hat{a}^{\mu\nu}\nabla_{\nu}\delta\hat{v}]\}+y \tag{13}$$

[$j_{\mu}^{(0)}$ is the equilibrium value of the chiral current, i.e., the solution of Eq. (7)].

An analysis of Langevin equation (12) or (13), e.g., a calculation of the correlation functions of the fluctuations $\delta\hat{v}$, can be formulated as the evaluation of a path integral with an effective action:[6]

$$S_{\text{eff}}=\int d^{2}xdt\{-p[\partial_{t}\delta\hat{v}-\hat{M}\delta\hat{v}+(1/2\gamma)p]+\bar{\psi}(\partial_{t}-\hat{M})\psi\}. \tag{14}$$

Here $\hat{M}$ is expressed in terms of the chiral currents (*i.e.*, it is determined by the internal geometry of the membrane. For the linear case, for example, we would have

$$\hat{M}=\gamma\{\partial_{\mu}(\hat{a}^{\mu\nu}\nabla_{\mu}\delta\hat{v})+[\nabla_{\mu}\delta\hat{v},\hat{a}^{\mu\nu}\hat{j}_{\nu}^{(0)}]+[\hat{j}_{\mu}^{(0)},\hat{a}^{\mu\nu}\nabla_{\mu}\delta\hat{v}]\}.$$

In (14), the quantity p is an auxiliary Bose variable, and ψ and $\bar{\psi}$ are anticommuting auxiliary fields.

Direct substitution easily shows that action (14) is supersymmetric, i.e., is invariant under the transformation

$$\delta\hat{v}=\gamma(\bar{\epsilon}\psi+\bar{\psi}\epsilon),\quad \delta\psi=\epsilon(\partial_{t}-\hat{M})\hat{v},\quad \delta\bar{\psi}=(\partial_{t}-\hat{M})\hat{v}\epsilon,$$

where ϵ and $\bar{\epsilon}$ are two anticommuting infinitesimal fields.

One can also verify that action (14) leads to the standard relation for relaxation dynamics:

$$-i\partial_{t}\langle\delta\hat{v}(0)\delta\hat{v}(t)\rangle=\langle p(0)\delta\hat{v}(t)\rangle-\langle p(t)\delta\hat{v}(0)\rangle. \tag{15}$$

This is the fluctuation dissipation theorem, since the correlation functions on the right side of (15) differ in no way from generalized susceptibilities or response functions of the membrane to a shape perturbation.[5]

4. The procedure described above reduces the calculation of arbitrary correlation functions characterizing the properties of membranes to a standard problem of chiral field theory with effective action (14). Under some very general assumptions, fluctuation dissipation theorem (15) has been established within the framework of the formalism proposed here. It has been proved that the effective action is supersymmetric.

It is important to note that the chiral currents in this theory retain information on the geometry of the surface even in such characteristics (which look quite local at first glance) as the correlation function of small fluctuations, $\langle \delta\hat{v}(1)\delta v(2)\rangle$.

Chiral field theory can also be used to calculate such nontrivial characteristics of a surface as the correlation functions of the mean or Gaussian curvature. From (6) and (10), for example, we have

$$\langle \delta k_h(1)\delta k_h(2)\rangle = \langle \delta[\hat{j}^1_{31}(1)+\hat{j}^2_{32}(1)]\delta[\hat{j}^1_{31}(2)+\hat{j}^2_{32}(2)]\rangle$$
$$= \langle [\nabla_1\delta\hat{v}^{31}(1)+\nabla_2\delta\hat{v}^{32}(1)][\nabla_1\delta v^{31}(2)+\nabla_2\delta v^{32}(2)]\rangle. \qquad (16)$$

In the planar case, the covariant derivatives in (16) reduce to ordinary derivatives, and all the calculations can be carried out completely. In the Fourier representation (ω is the frequency, and q the wave vector), we find from (14)

$$\langle \delta k_h(q,\omega)\delta k_h(0,0)\rangle = \frac{\gamma q}{\omega^2+\kappa^2\gamma^2 q^6}. \qquad (17)$$

This expression is the same as that derived in Ref. 5 for the correlation function for displacements of a membrane.

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