Matched differential geometry on the quantum groups $GL_q(2, C)$ and $SL_q(2, C)$

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A matched differential geometry is constructed on the quantum groups $GL_q(2,C)$ and $SL_q(2,C)$. The vector-field algebra is found by requiring that the quantum determinant commute with right and left 1-forms.

The studies of differential geometry on quantum groups by Woronowicz[1] and Wess and Zumino[2,3] made it clear that, in contrast with the situation for classical groups, there exists a set of differential geometries for quantum groups. Several efforts[4–9] have now been made to reduce the number of different differential geometries. These studies have been based on an effort to retain certain features of classical groups and of differential geometry on them: the simplicity of deformed commutation relations between the parameters of a group and their differentials and the bicoinvariance with respect to left and right multiplication on the group, among others. Different differential geometries on a quantum group lead to different quantum algebras for the vector fields, not all of them of the Drinfeld–Jimbo form.[10,11] Our purpose in the present letter is to construct a matched differential geometry on the quantum groups $GL_q(2, C)$ and $SL_q(2, C)$.

The quantum group $GL_q(2, C)$ is defined by commutation relations between elements of the group:

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \begin{array}{lll} ab=q\,ba, & bc=cb, & cd=q\,dc, \\ ac=q\,ca, & bd=q\,db, & ad-da=\left(q-\frac{1}{q}\right)bc, \end{array} \tag{1}$$

where q is a complex parameter of the deformation of the commutation relations.

An important element is the quantum determinant

$$D\equiv \det_q g=ad-qbc=da-\frac{1}{q}\,bc, \tag{2}$$

which is equal to 1 for the $SL_q(2, C)$ group. This determinant commutes with all elements of the group: $D\cdot g=g\cdot D$.

We wish to construct a quantum algebra. To do this we need to introduce an infinitely small neighborhood δg of the unity of the group, and we need to determine the commutation relations between the parameters of the group and the differentials. Equivalently, we need to determine the commutation relations between the parameters of the group and 1-forms and also those between 1-forms. We will be dealing with right forms $\omega=\delta g g^{-1}$ and left forms $\theta=g^{-1}\delta g$ simultaneously.

We find the commutation relations by requiring that the following relations hold:

$$\delta D = D\mathrm{Tr}_q\theta, \quad \delta D = \mathrm{Tr}_q\omega\, D,$$
$$D\theta = \theta\, D, \qquad D\omega = \omega\, D, \tag{3}$$

where $\mathrm{Tr}_q\omega$ and $\mathrm{Tr}_q\theta$ are the quantum traces of the matrices

$$\omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix},$$

which are still to be determined. We use the ordinary external differentiation operator δ, which satisfies Leibnitz's rule $[\delta(f\cdot g) = \delta f\cdot g + f\cdot \delta g)]$, and the conditions $\theta_k^2 = 0$, $\omega_k^2 = 0$, $k = 1, 2, 3, 4$. For the $SL_q(2, C)$ group we have $D = 1$ and therefore $\mathrm{Tr}_q\theta = \mathrm{Tr}_q\omega = 0$.

The most common type of commutation relation for left forms and for the parameters of the group which is matched with (3) is

$$\begin{aligned}
&\theta_1 d = A\, d\theta_1 + B\, d\theta_4, && \theta_4\, d = \tilde{A}\, d\theta_4 + \tilde{B}\, d\theta_1,\\
&\theta_1 a = F\, a\theta_1 + P\, a\theta_4, && \theta_4\, a = \tilde{F}\, a\theta_4 + \tilde{P}\, a\theta_1,\\
&\theta_2 d = q\, d\theta_2 + X\, c\theta_1 + \tilde{X}\, c\theta_4, && \theta_3\, d = q\, d\theta_3 + N\, c\theta_1 + \tilde{N}\, c\theta_4,\\
&\theta_2 a = \frac{1}{q}\, a\theta_2 + N\, b\theta_1 + \tilde{N}\, b\theta_4, && \theta_3 a = \frac{1}{q}\, a\theta_3 + Z\, d\theta_1 + \tilde{Z}\, a\theta_1,
\end{aligned} \tag{4}$$

where $A - Nq = 1 + \tilde{P} - N/q$, $F - \tilde{N}/q = 1 + B - \tilde{N}q$, and $\delta D = D[(A - Nq)\theta_1 + (F - \tilde{N}/q)\theta_4]$. The other relations are found by making the interchanges $d \leftarrow \rightarrow c$, $a \leftarrow \rightarrow b$, with the same coefficients.

The requirement that θ_k commute with commutation relations (1) leads to some additional equations for the coefficients; we will not reproduce those lengthy equations here. In the simplest version which leads to a Drinfeld–Jimbo algebra, we have $X = \tilde{X} = N = \tilde{N} = Z = \tilde{Z} = 0$, and the auxiliary equations are satisfied identically, except for the equations

$$\begin{aligned}
&P\tilde{P} - B\tilde{B} = 0, && AF + P\tilde{P} = 1, && FB + P\tilde{F} = 0,\\
&AP + \tilde{A}B - FB - P\tilde{F} = 0, && \tilde{A}F + P\tilde{P} = 1, && \tilde{A}\tilde{P} + \tilde{B}A = 0,\\
&\tilde{A}\tilde{P} + A\tilde{B} - \tilde{F}\tilde{B} - \tilde{P}F = 0.
\end{aligned} \tag{5}$$

To pursue the analysis we need to examine the $SL_q(2, C)$ group. In this case we have $\theta_4 + \Delta\theta_1 = 0$, where Δ is a parameter:

$$\begin{aligned}
&\theta_1 d = A d\theta_1, && \theta_2 d = q d\theta_2, && \theta_3 = q d\theta_3,\\
&\theta_1 a = F a\theta_1, && \theta_2 a = \frac{1}{q}\, a\theta_2, && \theta_3 a = \frac{1}{q}\, a\theta_3,\\
&\delta D = (A\theta_1 + \theta_4) = 0, && A = \Delta, && AF = 1,\\
&\delta D = (\theta_1 + F\theta_4) = 0, && F = \frac{1}{\Delta}.
\end{aligned} \tag{6}$$

Differentiating commutation relations (6), and using the Maurer–Cartan equation

$$\delta\Theta = -\Theta\Theta, \tag{7}$$

we find commutation relations between the elements of the θ-form and also the condition $A=q^2$:

$$\theta_2\theta_3+\frac{1}{q^2}\theta_3\theta_2=0,\quad \theta_1\theta_2+q^4\theta_2\theta_1=0,$$
$$\theta_1\theta_3+\frac{1}{q^4}\theta_3\theta_1=0,\quad \theta_k D=D\theta_k,\quad \delta D=D(q^2\theta_1+\theta_4)=D\mathrm{Tr}_q\theta=0. \tag{8}$$

Returning to the $GL_q(2, C)$ group, we find that the following condition must hold if commutation relations (4) are to be matched with (6), (8):

$$\tilde{A}=\frac{A}{q^2}.$$

Along with conditions (5) we find a one-parameter family of commutation relations:

$$\theta_1 d=Ad\theta_1+(\tilde{A}-1)d\theta_4,\qquad \theta_4 d=\left(1-A+\frac{A}{\tilde{A}}\right)d\theta_4+\frac{A(1-A)}{\tilde{A}}d\theta_1,$$
$$\theta_1 a=\left(1-\tilde{A}+\frac{\tilde{A}}{A}\right)a\theta_1+\tilde{A}\cdot\frac{1-\tilde{A}}{A}\cdot a\theta_4,\quad \theta_4 a=\tilde{A}a\theta_4+(A-1)a\theta_1,$$
$$\theta_2 d=qd\theta_2,\quad \theta_3 d=qd\theta_3,$$
$$\theta_2 a=\frac{1}{q}a\theta_2,\quad \theta_3 a=\frac{1}{q}a\theta_3,\qquad A=\tilde{A}q^2. \tag{9}$$

A further analysis of the algebra of vector fields shows that $\tilde{A}$ is an unimportant parameter and that for simplicity we can set $\tilde{A}=1$. We then need to supplement commutation relations (8) with some relations found by differentiating relations (9) and using the Maurer–Cartan equations:

$$\theta_2\theta_4+\theta_4\theta_1=0$$
$$\theta_4\theta_3+\theta_3\theta_4=q^2(q^4-1)\theta_1\theta_3,\quad \mathrm{Tr}_q\theta=q^2\theta_1+\theta_4. \tag{10}$$
$$\theta_4\theta_2+\theta_2\theta_4=q^2(q^4-1)\theta_2\theta_1,$$

A corresponding analysis for the right forms ω leads to the following commutation relations for the $SL_q(2, C)$ group:

$$d\omega_1=\frac{1}{q^2}\omega_1 d,\quad d\omega_2=\frac{1}{q}\omega_2 d,\quad d\omega_3=\frac{1}{q}\omega_3 d,$$
$$a\omega_1=q^2\omega_1 a,\quad a\omega_2=q\omega_2 a,\quad a\omega_3=q\omega_3 a,$$
$$\omega_2\omega_3+q^2\omega_3\omega_2=0,$$

$$\omega_1\omega_2+\frac{1}{q^4}\omega_2\omega_1=0,$$

$$\omega_1\omega_3+q^4\omega_3\omega_1=0,$$

$$\omega_k D=D\omega_k,\quad \delta D=\left(\frac{1}{q^2}\omega_1+\omega_4\right)D=0. \tag{11}$$

Other relations are found by making the interchanges $d\leftarrow\rightarrow c$, $a\leftarrow\rightarrow b$.

A single-parameter family of commutation relations for the GL(2, C) group is

$$d\omega_1-A\omega_1 d+q^2(1-\tilde{A})\omega_4 d,\quad d\omega_4=\tilde{A}\omega_4 d+\frac{1}{q^2}(1-A)\omega_1 d,$$

$$a\omega_1=q^2\tilde{A}\omega_1 a+q^4(\tilde{A}-1)\omega_4 a,\quad a\omega_4=q^2A\omega_4 a+(A-1)\omega_1 a, \tag{12}$$

$$A+\tilde{A}=\frac{q^2+1}{q^2}.$$

Again, the choice $\tilde{A}=1$ leads to commutation relations (11), supplemented by the conditions

$$\omega_1\omega_4+\omega_4\omega_1=0,$$

$$\omega_3\omega_4+\omega_4\omega_3=\frac{q^4-1}{q^2}\omega_3\omega_1,\quad \begin{matrix}\omega_k D=D\omega_k,\\ \mathrm{Tr}_q\omega=\frac{1}{q^2}\omega_1+\omega_4\end{matrix} \tag{13}$$

$$\omega_2\omega_4+\omega_4\omega_2=\frac{q^4-1}{q^2}\omega_1\omega_2.$$

By definition, the effect of applying a differential to an arbitrary function on the group is

$$\delta f=f\nabla_k^L\theta^k=\omega^k\nabla_k^R f, \tag{14}$$

where ∇^L and ∇^R are respectively left and right vector fields on the group.

From the condition $\delta^2 f=0$ and the Maurer–Cartan equations we find a vector-field algebra for the quantum group $SL_q(2, C)$:

$$\nabla_3^L\nabla_3^L-q^2\nabla_2^L\nabla_3^L=\nabla_1^L,\qquad \nabla_3^R\nabla_2^R-\frac{1}{q^2}\nabla_2^R\nabla_3^R=\nabla_1^R,$$

$$q^2\nabla_1^L\nabla_3^L-\frac{1}{q^2}\nabla_3^L\nabla_1^L=(q^2+1)\nabla_3^L,\quad \nabla_1^R\nabla_3^R-q^4\nabla_3^R\nabla_1^R=(q^2+1)\nabla_3^R, \tag{15}$$

$$q^2\nabla_2^L\nabla_1^L=\frac{1}{q^2}\nabla_1^L\nabla_2^L=(q^2+1)\nabla_2^L,\quad \nabla_2^R\nabla_1^R-q^4\nabla_1^R\nabla_2^R=(q^2+1)\nabla_2^R.$$

Here ∇_1^L represents the combination $\nabla_1^L-q^2\nabla_4^L$, and ∇_1^R represents the combination $\nabla_1^R-1/q^2\,\nabla_4^R$.

After the mapping

$$\nabla_1^L=\frac{q^2}{q^2-1}(1-q^{-2H}),\quad \nabla_2^L=q^{-H/2}T_2,\quad \nabla_3^L=q^{-H/2}T_3,$$
$$\nabla_1^R=\frac{1-q^{2H}}{1-q^4},\qquad \nabla_2^R=q^{-H/2}T_2,\quad \nabla_3^R=q^{-H/2}\,T_3, \tag{16}$$

we obtain the algebra $U_qSL(2, C)$ in the form of a Drinfeld–Jimbo algebra:

$$[T_3,T_2]=\frac{q^H-q^{-H}}{q-q^{-1}},\quad [H,T_3]=2T_3,\quad [H,T_2]=-2T_2.$$

The algebra of vector fields for the $GL_q(2, C)$ group has, in addition to (15), the following commutation relations:

$$[\nabla_1^L,\nabla_4^L]=0,\qquad [\nabla_1^R,\nabla_4^R]=0,$$
$$[\nabla_4^L,\nabla_2^L]=\nabla_2^L,\quad [\nabla_4^R,\nabla_2^R]=\nabla_2^R,$$
$$[\nabla_3^L,\nabla_4^L]=\nabla 3L,\quad [\nabla_3^R,\nabla_4^R]=\nabla_3^R.$$

We note in conclusion that the commutation relations between the parameters of the group and the differentials which follow from Eqs. (9) and (12) are different. This difference distinguishes the differential geometry constructed here from a bicoinvariant one. One can construct a matched bicoinvariant differential geometry for the quantum group $SL_q(2, C)$ without imposing the conditions $N=X=Z=0$. In this case, however, the vector-field algebra cannot be reduced to a Drinfeld–Jimbo algebra.

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