Local and global quantum effects on cosmic string space-time

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The effective action and vacuum energy of a quantum scalar field near a cosmic string are considered. Analogy is pointed out with quantum theory with boundaries. The surface infinities in the effective action are shown to appear and to be removed by renormalization of the string tension. The total renormalized energy of the string and field turns out to be finite due to cancellation of the known nonintegrable divergence in the energy density of the field with a counterterm in the bare string tension.

1. Quantum field theory on cosmic string space-time has recently attracted much attention,[1–6] because of its possible relevance in cosmology.[7] The space near an idealized cosmic string of zero thickness has interesting properties. It looks like a conical space in which the line element can be written in a form like that on the plane in polar coordinates, $ds^2=dr^2+r^2d\varphi^2$, but with a polar angle φ which ranges from 0 to a positive parameter $\alpha=2\pi(1-4\mu G)$, where μ is the string tension.[7,8] This space can also be considered as a space whose curvature is concentrated at the zero radius $r=0$, where it looks like a delta function.[8]

In connection with conical singularities, the quantum theory on orbifold factors of the Riemannian manifolds[9–11] is also worth mentioning. Singularities of that sort appear at fixed points of the corresponding isometry groups.

In the present paper we investigate the local and global effects of vacuum polarization near a cosmic string and we demonstrate a close analogy with quantum theories on spaces with boundaries.[12,13] The global effects are displayed in integral quantities like the ground energy or the effective action of the field. They can be derived by using the trace of the heat kernel on a cone which was shown to look fundamentally different at asymptotically small values of the proper time as compared with the plane heat kernel.[14] For this reason, the effective action obtained on its base includes a surface divergent functional given on the string world sheet. It is interesting that these surface infinities can be removed by renormalization of the string tension, which renders finite the total renormalized energy. A similar situation takes place in quantum theories with boundaries, where the divergent terms arise on boundary surfaces, which gives rise to renormalization of bare surface gravitational actions.[12] The analogy can be continued further to demonstrate that finiteness of the total renormalized energy results from cancellation of the nonintegrable divergence in the energy density with a surface counterterm appearing from the bare string tension.

2. For simplicity we restrict the analysis to the case of a free scalar field of mass m near an infinitely thin, straight string which is at rest along the z axis. The metric near it can be written in the form $ds^2=dt^2-dz^2-dr^2-r^2d\varphi^2$, where $0\leqslant\varphi\leqslant\alpha$.

The vacuum energy $E_0(\alpha)$ of the field can be calculated in two ways. The first involves obtaining $E_0(\alpha)$ as the integral of the renormalized energy density $\langle\hat{T}_{00}(x)\rangle^{\alpha}_{\text{sub}}$. However, this method cannot be applied directly since the renormalized energy momentum tensor has a nonintegrable infinity at the string axis[3-5] and an additional regularization is needed, as will be shown below. Let us first consider another method which is based on the thermodynamic relationship between the internal energy $E_{\beta-1}$ of the system at a temperature β^{-1} and the partition function Z_β:

$$E_{\beta^{-1}}=\langle\hat{H}\rangle_\beta=-\frac{\partial}{\partial\beta}\log Z_\beta\,,\quad Z_\beta=\text{Tr}(e^{-\beta\hat{H}}),\tag{1}$$

where $\hat{H}$ is the Hamiltonian. In such approach the ground energy E_0 is the energy at zero temperature, $E_0=\lim_{\beta\to\infty}E_\beta{}^{-1}$. This energy can be derived without using the renormalized energy density.

Another relevant quantity is the effective action W, which can be defined for a finite temperature, with the help of the partition function

$$W=-\log Z_\beta\tag{2}$$

by taking the limit $\beta\to\infty$. Its variations coincide with the thermal average of the variations of the functional

$$S_e[\phi]=\int\sqrt{g}\,d^4x1/2(\partial_\mu\phi\partial^\mu\phi+m^2\phi^2),$$

which are interpreted as quantum operators

$$\delta W=Z_\beta^{-1}\int D\phi\,\delta S_e\,[\phi]e^{-S_e[\phi]}=\langle\delta\hat{S}_e\rangle_\beta\,,\quad Z_\beta=\int D\phi e^{-S_e[\phi]}.\tag{3}$$

We take account in (3) of the fact that the partition function can be represented in the form of a functional integral by changing to a periodic imaginary time. Consequently, the action $S_e[\phi]$ is taken here on the Euclidean section of the string space-time $ds^2=d\tau^2+dz^2+dr^2+r^2d\varphi^2$, $0\leqslant\tau\leqslant\beta$. The action W is therefore a Euclidean form of the effective action, and a transition to the convenient definition[13,15] does not create any difficulties for the static space we are considering.

If we represent the action W in the form

$$W=\frac{1}{2}\log\det\,(\Delta+m^2)=\frac{1}{2}\,\text{Tr}\,\log(\Delta+m^2)=-\frac{1}{2}\int_0^\infty\frac{ds}{s}\,\text{Tr}(e^{-s\Delta})e^{-m^{2S}},\tag{4}$$

where Δ is the corresponding Laplace operator, then the structure of its ultraviolet divergences follows directly from the asymptotic expansion, as $s\to 0$, of the trace $\text{Tr}(e^{-s\Delta})$ of the heat kernel on the Euclidean string space-time[14]

$$\text{Tr}(e^{-s\Delta})=\frac{1}{(4\pi s)^2}\,[\Omega+\Sigma\alpha C_1(\alpha)s]+ES,\quad(\beta\to\infty),\tag{5}$$

where $C_1(\alpha)=1/6[(2\pi\alpha^{-1})^2-1]$, ES are the terms which vanish exponentially as $s\to 0$, and Ω is the volume of the space. The second term in (5) is contributed solely by the conical singularities on the string. This term is proportional to the area Σ of the surface $r=0$, the string world sheet. It is important that this term does not appear when the integration over the space-time in the effective action is stopped short before the point $r=0$, no matter how close to it.

As long as the space is noncompact, Ω and Σ are infinite, and (5) therefore must be treated in a regularized manner. In this case the ES terms are important. However, if L is a typical size of the space (the length at which the integrals are cut off), then the ES terms in (5) can be shown to be on the order of $s^{-1}\exp(-L^2/s)$. We can therefore drop ES, which are negligible small terms in the limit $L\to\infty$ which we are considering.

After substitution of (5) into (4) we can represent the effective action as a sum of the volume and surface parts which are proportional to Ω and Σ, respectively: $W=W_{\rm vol}+W_{\rm surf}$. The volume term $W_{\rm vol}$ in the action turns out to be exactly the same as in Minkowsky space and it develops standard divergences which appear at the lower integration limit in (4) as $s\to 0$. In addition, the conical singularities at the string axis account for the additional divergent surface term $W_{\rm surf}$, whose exact form is not relevant for this analysis. Since the latter is proportional to the area Σ of the string world sheet, this term should be unified with the string action which can be added to W and the divergence can be removed by renormalizing the bare string tension μ_B. With the help of this recipe we can obtain the total renormalized effective action

$$W_{\rm tot}=W+\mu_B\sum=W_{\rm vol}+\mu\sum, \tag{6}$$

where the infinities in $W_{\rm vol}$ are removed by a standard procedure, and the string action is expressed in terms of the renormalized tension μ. [In (6) we used the fact that W and $W_{\rm tot}$ are the actions on the space-time with a Euclidean signature which results in the string action in the form $\mu\Sigma$, where $\Sigma=\beta\int dz$ and $\beta\to\infty$.]

Taking into account (1) and (2), we can write the vacuum energy as $E_0(\alpha)=\partial/\partial\beta W$ (as $\beta\to\infty$). However, W is a divergent functional. Consequently, a finite result can be obtained for the total energy $E_{\rm tot}=\partial/\partial\beta W_{\rm tot}$, which is defined by the renormalized total action and which includes the energy of the string. After subtracting from $E_{\rm tot}$ the vacuum energy in the Minkowsky space, which is equivalent to omitting the volume part $W_{\rm vol}$ in $W_{\rm tot}$, we obtain

$$E_{\rm tot}=\mu\int dz. \tag{7}$$

It shows that the total renormalized energy per unit length turns out to be finite and it is determined only by the renormalized string tension μ.

3. Until now, we dealt with the integral quantities, like the effective action and total energy, using for their calculation the trace of the heat kernel. The surface terms appearing in the functional W have a global origin: They would have not arisen if we had excluded from the integrals over the space-time the region around the string world sheet. The local renormalized energy momentum tensor near the cosmic string was calculated by a number of authors.[3–5] Let us determine the connection between their results and our results and demonstrate that the local nonintegrable divergence in the average energy density, which arises as the string is approached, can be removed by a suitable renormalization of the bare string tension, so that the total energy turns out to be finite. We are going to explore the same approach as that used in quantum theory with boundaries.

We consider a real, massless, scalar field for which the energy density can be obtained in the closed form[3]

$$\langle \hat{T}_{00}(x)\rangle^{\alpha}_{\text{sub}} = \frac{1}{16\pi^2 r^4}\,[2(1-4\xi)C_1(\alpha) - C_2(\alpha)], \tag{8}$$

where $C_2(\alpha) = 1/90[(2\pi\alpha^{-1})^2 - 1][(2\pi\alpha^{-1})^2 + 11]$. The value $\xi = 1/6$ corresponds to a conformally invarient field. The local energy is evaluated in a standard way from the Green's function $G^{\alpha}(x,x') = i^{-1}\langle T[\hat{\phi}(x),\hat{\phi}(x')]\rangle$ as a coincidence limit $\langle \hat{T}_{00}(x)\rangle^{\alpha}_{\text{sub}} = \lim_{x'\to x}\mathscr{T}_{00}G^{\alpha}_{\text{sub}}(x,x')$, where $\mathscr{T}_{00}$ denotes a second-order differential operator[16] which depends on the type of field, and the divergences are removed in the ordinary way by subtracting the Minkowsky Green's function $G^{\alpha=2\pi}$ in G^{α}_{sub}.

It is obvious that the local divergence of the energy density (8) at $r=0$ can be regularized if we restrict the domain of integration in the total energy by the values of coordinates $r \geqslant r_0$, where r_0 is a positive small parameter which can be treated as the string radius. Moreover, the regularization which we suggested also makes finite the surface term in the effective action. This can be demonstrated for the particular values of the parameter $\alpha = 2\pi n^{-1}$, $n = 2, 3, \ldots$, when the heat kernel $K_\alpha(r, r', \varphi - \varphi')$ on the conical space can be represented in a closed form with the help of the plane heat kernel $K(r,r',\varphi-\varphi') = (4\pi s)^{-1}\exp\{-[r^2 + r'^2 - 2rr'\cos(\varphi-\varphi')]/4s\}$. For example, for $\alpha = \pi$ we can write $K_\alpha(r, r', \Delta\varphi) = K(r, r', \Delta\varphi) + K(r, r', \Delta\varphi + \pi)$. In this case the "regularized" trace can be easily calculated and we obtain instead of (5) the expression

$$\operatorname{Tr}(e^{-s\Delta})_{r_0} \equiv \int_{r_0}^{\infty} r\,dr \int_0^{\alpha} d\varphi \int d\tau dz\, \frac{1}{4\pi s} K_\alpha(r, r, 0) = \frac{1}{4\pi s}\left(\Omega + \frac{\pi s}{2} e^{-r_0^2/s}\Sigma\right). \tag{9}$$

The surface term in the trace now does not produce a divergence in the effective action (4) after integration over s. The total effective action can be defined as before. Here it takes the form

$$W_{\text{tot}} = \frac{1}{2}\log\det(\Delta + m^2)_{r_0} + \mu_B\Sigma = -\frac{1}{2}\int_0^{\infty}\frac{ds}{s}\operatorname{Tr}(e^{-s\Delta})_{r_0} e^{-m^2 s} + \mu_B\Sigma$$

$$\equiv W_{\text{vol}} + W_{r_0,\,\text{surf}}\,. \tag{10}$$

It is separated into the volume part W_{vol}, which is proportional to Ω, and the surface part $W_{r_0,\text{surf}}$ which is given on the world sheet Σ. In contrast with W_{vol} which develops the standard divergences, the surface action $W_{r_0,\text{surf}}$ turns out to be finite and $r_0 \neq 0$. In the case of arbitrary values of α and for zero mass its expression can be determined exactly with the help of an integral representation[1,2] of the kernel K_α:

$$W_{r_0,\text{surf}}=\left(\mu_B-\frac{\alpha C_2(\alpha)}{32\pi^2 r_0^2}\right)\Sigma. \tag{11}$$

It follows from Ref. 11 that the divergence in $W_{r_0,\text{surf}}$ in the limit $r_0\to 0$ can be removed by replacing, as before, the bare string tension μ_B by the renormalized μ

$$\mu_B=\mu+\frac{1}{32\pi^2 r_0^2}\alpha C_2(\alpha). \tag{12}$$

Taking this fact into account, we can write the local renormalized energy as the sum

$$\langle\hat{T}_{00}\rangle^{\alpha}_{r_0,\text{ren}}=T_{00,B}+\langle\hat{T}_{00}\rangle^{\alpha}_{r_0,\text{sub}} \tag{13}$$

of the string energy $T_{00,B}=\mu_B\delta_2(r)$, which is concentrated at the string axis [$\delta_2(r)$ is the delta function on a cone

$$\int_0^\infty rdr\int_0^\alpha d\varphi\delta_2(r)=1],$$

and the renormalized energy density $\langle\hat{T}_{00}\rangle^{\alpha}_{r_0,\text{sub}}$ of the quantum field in the domain $r\geqslant r_0$. Two densities, $\langle\hat{T}_{00}\rangle^{\alpha}_{\text{sub}}$ given by (8) and $\langle\hat{T}_{00}\rangle^{\alpha}_{\tau_0,\text{sub}}$ coincide everywhere except in the region near the string. To demonstrate this point, let us calculate the classical energy-momentum tensor of the field in this domain which is defined by the functional differentiation of the action which we take in the same form as in Ref. 12

$$S=-\frac{1}{2}\int_{\tau\geqslant\tau_0}d^4x\sqrt{-g}\,\phi(x)[\Box+\xi R]\phi(x), \tag{14}$$

where $\Box=\sqrt{-g}^{\,-1}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$ is the D'Alambertian, and R is the scalar curvature. The variation of this functional δS as a result of changing the metric $\delta g^{\mu\nu}$ consists of two parts:

$$\delta S=\frac{1}{2}\int_{\tau\geqslant\tau_0}d^4x\,\sqrt{-g}\,T_{\mu\nu}(x)\delta g^{\mu\nu}(x)+\delta_{\text{surf}}S, \tag{15}$$

where $T_{\mu\nu}$ stands for the normal expression of the stress tensor of a scalar field[17] and an additional surface term arises due to the restriction of the domain of integration

$$\delta_{\text{surf}}S=-\frac{1}{2}\int_{\tau=\tau_0}d\sigma^\tau\{\phi^2(g_{\mu\nu}\delta g^{\mu\nu};\ \tau+g_{\tau\mu}\delta g^{\mu\sigma};\ \sigma)$$
$$+[(1/4-\xi)(\phi^2),\ _\tau g_{\mu\nu}+(\xi-1/2)(\phi^2),\ _\nu g_{\mu\tau}]\delta g^{\mu\nu}\} \tag{16}$$

($d\sigma^\tau$ is the area element). As long as there is no real boundary of the space on the surface $r=r_0$, the variations of the metric $\delta g^{\mu\nu}|_{\tau=\tau_0}$ do not vanish on it. They are independent of its normal derivatives on the surface and thus the last ones can be ignored. As a result, $\delta_{\text{surf}}S$ produces an additional term in the energy density

$$T_{00,\mathrm{surf}}=\frac{2}{\sqrt{-g}}\frac{\delta_{\mathrm{surf}}S}{\delta g^{00}}=(1/4-\xi)\delta(r-r_0)\frac{d}{dr}(\phi)^2, \tag{17}$$

which accounts for the distinction between the average density in the domain, $\langle\hat{T}_{00}\rangle^{\alpha}_{r_0,\mathrm{sub}}$, and the local energy (8)

$$\langle\hat{T}_{00}\rangle^{\alpha}_{r_0,\mathrm{sub}}=\langle\hat{T}_{00}\rangle^{\alpha}_{\mathrm{sub}}+i(1/4-\xi)\delta(r-r_0)\lim_{x\to x'}\left(\frac{d}{dr}+\frac{d}{dr'}\right)G^{\alpha}_{\mathrm{sub}}(x,x') \tag{18}$$

[$\delta(r-r_0)$ is the one-sided delta-function]. For its calculation we can use the proper-time representation for the Green's function. The final result is[14]

$$\langle\hat{T}_{00}\rangle^{\alpha}_{r_0,\mathrm{sub}}=\langle\hat{T}_{00}\rangle^{\alpha}_{\mathrm{sub}}-(1/4-\xi)\frac{C_1(\alpha)}{4\pi r_0^3}\delta(r-r_0). \tag{19}$$

Integrating the renormalized quantity (13) over the space

$$E_{\mathrm{tot}}=\int\langle\hat{T}_{00}\rangle^{\alpha}_{r_0,\mathrm{ren}}\,dv=\left[\mu_B+\int_{r_0}^{\infty}rdr\int_0^{\alpha}d\varphi\langle\hat{T}_{00}\rangle^{\alpha}_{r_0,\mathrm{sub}}\right]\int dz \tag{20}$$

and using (12) and (20), we find that the counterterm in the bare tension μ_B cancels exactly the term proportional to r_0^{-2} in the integrated energy of the field, which renders finite the renormalized total energy as $r_0\to 0$:

$$E_{\mathrm{tot}}=\mu\int dz. \tag{21}$$

This expression coincides with the total energy (7) which was derived before in another way from W_{tot}.

4. In this study we established a close analogy between quantum theory on the space with conical singularities and quantum theory with boundaries. In each case the one-loop quantum corrections give divergent surface functionals in the effective actions. The renormalization of these functionals can be used to remove nonintegrable divergence in the energy density and to obtain the finite total energy of the system. However, this analysis concerns the idealized objects, strings, and boundaries of zero thickness. In effect, one might expect that for the real string of finite size the divergent terms on its world sheet give large but still finite contributions to the renormalized energy.

In the theory with boundaries the surface actions are known to depend essentially on which of the two boundary conditions, Dirichlet or Neumann, is imposed. As for the string case, Eq. (5) implies a finite boundary condition on the string axis.[14] Other possibilities should also be investigated. For example, the possibility for the existence of logarithmically divergent conditions has been pointed out in Ref. 6 in connection with the self-adjoint extensions of the Laplace operator on a cone.

It should also be mentioned that our analysis was restricted essentially to conical singularities in the flat space, and incorporation of the curvature effects is an interesting problem.

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