# Nonlinear kink oscillations of a magnetic flux tube

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The nonlinear equations which describe the long wavelength, weakly dispersive, kink oscillations propagating along a magnetic flux tube are derived. The character of nonlinearity appeared to be cubic, with the coefficients which reflect the influence of a magnetic free environment on the transverse oscillations of the flux tube.

In recent years the characteristic features of magnetic flux tubes have been studied extensively because of their dominant role in the dynamics of solar atmosphere: According to the observational data, magnetic field at the solar surface occurs not in a diffuse form but is concentrated in thin, intense, magnetic flux bundles embedded in nearly nonmagnetized plasma. Usually magnetic flux tubes are isolated and far removed from each other, covering 90% of the solar atmosphere outside sunspots.[1] In sunspot regions magnetic flux tubes form a dense conglomerate.[2] A structured magnetic field is often encountered in a laboratory plasma and in other astrophysical objects.

The interaction of magnetic flux tubes with the surrounding plasma results in the excitation of different kinds of oscillations which propagate along flux tubes.[3] Among them the most important one is a kink oscillation corresponding to the dipole mode with the azimuthal wave number $m=\pm 1$ and the phase velocity

$$c=\frac{B}{\sqrt{4\pi(\rho_i+\rho_e)}}. \tag{1}$$

Here $\rho_i$ and $\rho_e$ are plasma densities inside and outside the flux tube, and $B$ is the magnetic field strength. The linear oscillations of the flux tube are now well understood, and are discussed elsewhere[3] (see also Ref. 4 and the bibliography cited there).

In the present paper we derive the equations which govern the propagation of weakly nonlinear, long-wavelength kink mode which propagates along the magnetic flux tube surrounded by nonmagnetized plasma. We adopt the model of a cylindrical flux tube of radius $R$, which is assumed to be much smaller than the wavelength $\lambda=k^{-1}$:

$$kR\ll 1. \tag{2}$$

The discussion is based on the ideal MHD equations which are written here for convenience:

  

$$\rho\frac{\partial \mathrm{v}}{\partial t}+(\mathrm{v}\nabla)\vec{v}=-\nabla p+\frac{1}{4\pi}[\mathrm{curl}\,\mathbf{B},\mathbf{B}], \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t}=\mathrm{curl}[\mathrm{v}\mathbf{B}],\quad \frac{\partial p}{\partial t}+\mathrm{div}\,\rho\vec{v}=0,\quad p=p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}. \tag{4}$$

The above set of equations should be supplemented with the equation for the pressure balance in an equilibrium state of the flux tube:

$$p_i(r)+\frac{B^2(r)}{8\pi}=p_e. \tag{5}$$

Here $p_i$ and $p_e$ are gas kinetic pressures inside and outside the magnetic flux tube. For simplicity, we assume that the plasma inside the flux tube is cold, $p_i \ll p_e$, and, accordingly, we ignore the gas-kinetic pressure inside the flux tube. This assumption is not essential but it allows us to simplify the algebra. Equations (3) and (4) describe the motion inside the flux tube and outside it (where these equations become pure hydrodynamic equations). At the surface of the flux tube the boundary conditions of continuity of the normal component of the velocity,

$$v_{ri}|_{r=R}=v_{re}|_{r=R}, \tag{6}$$

and of the normal component of the momentum flux,

$$p_i+\frac{B^2}{8\pi}|_{r=R}=p_e|_{r=R}, \tag{7}$$

should be satisfied. We choose the cylindrical coordinate system whose $z$ axis is directed along the magnetic field.

According to the linear theory developed in Ref. 3 for perturbations, proportional to $\exp(-i\omega t+im\varphi+ikz)$, the MHD equations are reduced to a single equation for the function $\psi$ inside the flux tube,

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r}+\left[\frac{\omega^2}{v_A^2}-k^2-\frac{1}{r^2}\right]\psi=0, \tag{8}$$

and to a single equation for the velocity potential $\mathbf{v}=-\nabla\chi$ outside it,

$$\frac{1}{r}\frac{\partial}{\partial t}r\frac{\partial\chi}{\partial r}+\left[\frac{\omega^2}{c_{se}^2}-k^2-\frac{1}{r^2}\right]\chi=0. \tag{9}$$

Here $v_A$ is the Alfvén velocity, and $c_S=\sqrt{\gamma p/\rho}$ is the sound velocity outside the flux tube. The velocity and magnetic field perturbations are expressed in terms of the function $\psi$ as follows:

$$v_r=-\frac{\partial\psi}{\partial r},\quad v_\varphi=-\frac{1}{r}\frac{\partial\psi}{\partial\varphi},\quad v_z=0, \tag{10}$$

$$b_r=\frac{kB}{\omega}\frac{\partial\psi}{\partial r},\quad b_\varphi=\frac{im}{r}\frac{kB}{\omega}\psi,\quad b_z=-\frac{i\omega B}{c_{si}^2+v_A^2}\left[1-\frac{k^2v_A^2}{\omega^2}\right]\psi. \tag{11}$$

Accordingly, the pressure perturbations outside the flux tube is

$$\delta p_e = -i(\omega - ku)\rho_e \chi. \tag{12}$$

Inside the flux tube the solution is proportional to a first-order ($m=1$) Bessel function: $\psi = \mathscr{A}\mathscr{I}_1(q_i r)$ with $q_i^2 = (\omega^2 - k^2 v_A^2)/v_A^2$. In the outer region the solution of (9) should have the form of a divergent wave: $\chi = D\mathscr{H}_1^{(1)}(q_e r)$, where $\mathscr{H}$ is a Hankel function, and $q_e^2 = (\omega^2 - k^2 c_{se}^2)/c_{se}^2$.

The linearized boundary conditions (6) and (7) lead to the dispersion relation

$$\eta \frac{\omega^2 - k^2 v_A^2}{\omega^2} \frac{\partial \ln \chi}{\partial r} = \frac{\partial \ln \psi}{\partial r}, \tag{13}$$

where we use the notation $\eta = \rho_i/\rho_e$. In the long-wavelength limit (2) expression (13) is expanded in powers of a small parameter (kR). The first term in this expansion gives the phase velocity of the kink mode, (1). Retaining the next order terms, we obtain the dispersion relation

$$\omega = ck + \beta k^3 + i\mu_k^3, \tag{14}$$

where

$$\beta = -\frac{cR^2}{8(1+\eta)^2}, \quad \mu = \frac{\pi c R^2}{4} \frac{c^2 - c_s^2}{(1+\eta)c_s^2}. \tag{15}$$

Here the second term describes a weak dispersion of the kink mode, and the third term corresponds to the radiative damping of flux tube oscillations described in Ref. 3: According to this effect, the oscillation flux tube gives off its energy through the radiation of secondary acoustic waves.

To determine the character of the nonlinearity of the kink mode, we should take into account that the first nonlinear term which can affect the finite-amplitude kink oscillations is a cubic term, since the azimuthal dependence of the quadratic nonlinearity contains only the terms with $m=0$ and $m=2$. Taking this fact into account and using dispersion relation (14), we introduce the stretched variables

$$\zeta = \epsilon(z - ct), \quad \tau = \epsilon^2 t. \tag{16}$$

To carry out adequate perturbation expansion of the MHD equations, we represent the velocity and the magnetic field with power series expansion in $\epsilon$ as follows:

$$\begin{aligned}
\mathrm{v}_\perp &= \epsilon^{1/2}\mathrm{v}_{1\perp} + \epsilon^{3/2}\mathrm{v}_{2\perp} = \cdots, \\
v_z &= \epsilon v_{1z} + \epsilon^2 v_{2z} + \cdots, \\
\mathrm{B}_\perp &= \epsilon^{1/2}\mathrm{B}_{1\perp} + \epsilon^{3/2}\vec{B}_{2\perp} + \cdots, \\
B_z &= B_0 + \epsilon^{3/2}B_{1z} + \epsilon^{5/2}B_{2z} + \cdots, \\
\rho &= \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots.
\end{aligned} \tag{17}$$

Outside the flux tube we have, respectively,

$$
\begin{aligned}
&\mathbf{v}_{e\perp} = \epsilon^{1/2}\mathbf{v}_{e1\perp} + \epsilon^{3/2}\mathbf{v}_{e2\perp} + \cdots,\\
&v_{ez} = \epsilon^{3/2} v_{e1z} + e^{5/2} v_{e2z} + \cdots,\\
&p_e = p_{e0} + \epsilon^{3/2} p_{e1} + \epsilon^{5/2} p_{e2} + \cdots,\\
&\rho_e = \rho_{e0} + \epsilon^{3/2}\rho_{e1} = \epsilon^{5/2}\rho_{e2} + \cdots,
\end{aligned} \tag{18}
$$

where $\mathbf{v}_\perp$ and $\mathbf{B}_\perp$ are the transverse ($r$ and $\varphi$) components of the velocity and the magnetic field. The expansions (17) and (18) give the correct description of the linear stage of the flux tube oscillations and are consistent with the main features of weakly dispersive, long-wavelength, transverse oscillations of the flux tube which is embedded in the nonmagnetic region. It is important to note that in this limit the above choice allows us to specify the nature of the nonlinearity separately from the (weak) dispersion which has, in this limit, the form obtained from linear analysis. Substituting (17) and (18) in the MHD equations and equating terms of each order in $\epsilon$, we obtain a sequence of equations up to the desired order. First, for the order of $\epsilon^{3/2}$ we have from (3) and (4) the relations

$$
-c\rho_0 \frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta} = -\frac{B_0}{4\pi} \nabla_\perp B_z + \frac{B_0}{4\pi}\frac{\partial \mathbf{B}_{1\perp}}{\partial \zeta}, \tag{19}
$$

$$
-c \frac{\partial \mathbf{B}_{1\perp}}{\partial \zeta} = B_0 \frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta}, \tag{20}
$$

and for the outer region we have

$$
-c\rho_{e0} \frac{\partial \mathbf{v}_{e1\perp}}{\partial \zeta} = -\nabla_\perp p_1, \tag{21}
$$

$$
p_1 = c_s^2 \rho_1 . \tag{22}
$$

From (20) we have

$$
\mathbf{B}_{1\perp} = -\frac{B_0}{c} \mathbf{v}_{1\perp} . \tag{23}
$$

Substituting $\mathbf{B}_{1\perp}$ from (23) into (19), we obtain

$$
\left( -c\rho_0 + \frac{B_0^2}{4\pi c} \right) \frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta} = -\nabla_\perp \frac{B_0 B_{1z}}{4\pi}. \tag{24}
$$

We see from (24) that $\mathbf{v}_{1\perp}$, and consequently $\mathbf{B}_{1\perp}$ can be expressed in terms of some function $\psi$ exactly in the same way as above [cf. Eqs. (10) and (11)]. In other words,

$$
\mathbf{v}_{1\perp} = -\nabla_\perp \psi, \quad \mathbf{B}_{1\perp} = \frac{B_0}{c} \nabla_\perp \psi, \quad B_{1z} = \frac{B_0}{c} \frac{v_A^2 - c^2}{v_A^2} \frac{\partial \psi}{\partial \zeta}. \tag{25}
$$

Using the boundary conditions (6) and (7), we find

$$
\left( -c\rho_0 + \frac{B_0^2}{4\pi c} \right) \frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta} = -c\rho_{e0} \frac{\partial \mathbf{v}_{1\perp}}{\partial_\zeta}. \tag{26}
$$

This expression coincides with those obtained in Ref. 3 for the linear oscillations of the flux tube and gives the phase velocity such as (1). The validity of (26) can be easily shown by integrating the $r$-component of Eq. (24) over $r$ in the entire space. We represent the functions $\psi$ and $\chi$ as follows:

$$\psi = AX_i(r)e^{i\varphi}\xi_i(z,t), \quad \chi = DX_e(r)e^{i\varphi}\xi_e(z,t).$$

Integrating the $r$ component in Eq. (24) in the whole space with the help of (27), we obtain

$$\int_0^R \left(-c\rho_0 + \frac{B_0^2}{4\pi c}\right) A \frac{\partial X_i(r)}{\partial r} \frac{\partial \xi_i}{\partial \zeta} dr - \int_R^\infty c\rho_e D \frac{\partial X_e}{\partial r} \frac{\partial \xi_e}{\partial \zeta} dr$$

$$= \int_0^R \frac{\partial}{\partial r} \frac{B_0 B_{1z}}{4\pi} dr + \int_R^\infty \frac{\partial p_{1e}}{\partial r} dr. \tag{28}$$

The continuity of the momentum flux eliminates r.h.s. in Eq. (28), and the continuity of the normal component of the velocity leads directly to Eq. (26).

Next, for the order of $\epsilon^2$ we have

$$\rho_1 = \frac{\rho_0}{c} v_{1z}, \tag{29}$$

from the second equation in (4), and from the $z$ component in (3) we have

$$\rho_0 c v_{1z} = \frac{c^2}{v_A^2} \frac{B_{1\perp}^2}{8\pi}. \tag{29a}$$

Here we used relations (25) and their consequence in the form

$$\nabla_\perp B_{1z} = \frac{v_A^2 - c^2}{v_A^2} \frac{\partial \mathbf{B}_{1\perp}}{\partial \zeta}. \tag{30}$$

Now, at the order of $\epsilon^{5/2}$ the transverse components of Eq. (3) and of the first equation in (4) give

$$-c\rho_0 \frac{\partial v_{2\perp}}{\partial \zeta} - \frac{B_0}{4\pi} \frac{B_{2\perp}}{\partial \zeta} + \frac{B_0}{4\pi} \nabla_\perp B_{2z} = -\rho_0 \frac{\partial v_{1\perp}}{\partial \tau} + c\rho_1 \frac{\partial v_{1\perp}}{\partial \zeta} - \rho_0 v_{1z} \frac{\partial v_{1\perp}}{\partial \zeta} \tag{31}$$

and

$$c \frac{B_{2\perp}}{\partial \zeta} + B_0 \frac{\partial v_{2\perp}}{\partial \zeta} = \frac{\partial B_{1\perp}}{\partial \tau} + \frac{\partial}{\partial \zeta} (v_{1z} B_{1\perp}). \tag{32}$$

In the outer region of this order we have

$$-c\rho_{e0} \frac{\partial \mathbf{v}_{e2\perp}}{\partial \zeta} + \nabla_\perp p_2 = -\rho_{e0} \frac{\partial \mathbf{v}_{e1\perp}}{\partial \tau} - \frac{\rho_{e1}}{2} \nabla_\perp \mathbf{v}_{e1}^2, \tag{33}$$

$$-c\rho_{e0} \frac{\partial v_{e1z}}{\partial \zeta} = -\frac{\partial p_1}{\partial \zeta}, \tag{34}$$

or taking into account that $p_1 = c_s^2 \rho_{e1}$, instead of (34) we have

$$c\rho_{e0}v_{e1z}=\rho_{e1}c_s^2\,. \qquad (35)$$

Combining Eqs. (33) and (35) and taking into account that

$$\nabla\rho_{e1}=d\frac{c\rho_{e0}}{c_s^2}\frac{\partial \mathbf{v}_{e1\perp}}{\partial\zeta}, \qquad (36)$$

we can rewrite Eq. (33) in the form

$$-c\rho_{e0}\frac{\partial \mathbf{v}_{e2\perp}}{\partial\zeta}+\nabla_\perp\left(p_2+\frac{\rho_{e1}\mathbf{v}_{e1\perp}^2}{2}\right)=-\rho_{e0}\frac{\partial \mathbf{v}_{e1\perp}}{\partial\tau}+\frac{c\rho_{e0}}{c_s^2}\mathbf{v}_{e1\perp}^2\frac{\partial \mathbf{v}_{e1\perp}}{\partial\zeta}\,. \qquad (37)$$

Matching Eqs. (31) and (37) in terms of the boundary conditions, we obtain

$$c(\rho_{i0}+\rho_{e0})\frac{\partial \mathbf{v}_{2\perp}}{\partial\zeta}+\frac{B_0}{4\pi}\frac{\partial \mathbf{B}_{2\perp}}{\partial\zeta}=c(\rho_{i0}+\rho_{e0})\frac{\partial \mathbf{v}_{1\perp}}{\partial\tau}-c\frac{\rho_{e0}}{2c^2}\mathbf{v}_{1\perp}^2\frac{\partial \mathbf{v}_{1\perp}}{\partial\zeta}\,. \qquad (38)$$

Eliminating the second-order terms in Eqs. (38) and (32), we obtain straightforward nonlinear equations with respect to the stretched variables

$$2\frac{\partial \mathbf{B}_{1\perp}}{\partial\tau}+\frac{c}{\rho_{i0}v_A^2}\frac{\partial}{\partial\zeta}\left(\mathbf{B}_{1\perp}\frac{\mathbf{B}_{1\perp}^2}{8\pi}\right)-\frac{c^2}{c_s^2}\frac{c\rho_{e0}}{B_0^2(\rho_{i0}+\rho_{e0})}\frac{\mathbf{B}_{1\perp}^2}{2}\frac{\partial \mathbf{B}_{1\perp}}{\partial\zeta}\,. \qquad (39)$$

It is convenient to introduce instead of the transverse components $\mathbf{B}_\perp$ the complex quantity $H=B_r-iB_\varphi$ and normalize it by the unperturbed magnetic field $B_0$. Finally, the nonlinear equation for a kink mode can then be written as

$$\frac{\partial H}{\partial\tau}+\frac{c}{4}\frac{\partial}{\partial\zeta}(|H|^2H)-\frac{c^3}{4(1+\eta)c_s^2}|H|^2\frac{\partial H}{\partial\zeta}=0. \qquad (40)$$

Equation (40), aside from the last term, is similar to the equations obtained for hydromagnetic waves parallel to the magnetic field in a cold plasma.[6,7] We would like to emphasize that, in contrast with the case of an unbounded plasma considered in Refs. 6 and 7, here we are dealing with an oscillating magnetic string which interacts with the nonmagnetic surroundings. The influence of the magnetic free region is accounted for by the last term in Eq. (40) and is reflected in the propagation velocity of the kink mode [cf. Eq. (1)] which contains the plasma density outside the flux tube. Equation (40) should be supplemented with the dispersion and radiative damping terms obtained in a linear analysis. With the help of dispersion relation (14) a standard procedure gives the equation describing the evolution of a weakly nonlinear and weakly dispersive oscillations of the magnetic flux tube. It corresponds to the dipole (kink) mode and contains the condition under which the oscillating flux tube radiates secondary acoustic waves:

$$\frac{\partial H}{\partial\tau}+\frac{c}{4}\frac{\partial}{\partial\zeta}(|H|^2H)-\frac{c^3}{4(1+\eta)c_s^2}|H|^2\frac{\partial H}{\partial\zeta}+\beta\frac{\partial^3 H}{\partial\zeta^3}+\frac{\mu}{\pi}\,\mathrm{v.p.}\int_{-\infty}^{\infty}\frac{\partial^3 H}{\partial\zeta^3}\frac{ds}{\zeta-\zeta}=0, \qquad (41)$$

where $\beta$ and $\mu$ are given by expressions (15). The effect of radiative damping is very important in the study of the dynamics of a flux tube in the presence of shear mass flows along the magnetic field.[4] As it was shown in Ref. 4, negative energy waves (NEW) in this case can be excited along the magnetic flux tube. According to the main feature of the NEW, which consists in growing their amplitudes due to any kind of dissipation, the radiative damping provides the development of a strong instability in those regions where the NEW can be excited. In the next paper of the ongoing studies[8] we will derive the evolution equation similar to the equations obtained above in the presence of shear mass flow along the magnetic flux tube, and we will show that, depending on the parameters of the flux tube and on the surrounding plasma, a nonlinear equation such as (41), with shear flow as the source of energy, leads to vigorous nonlinear dynamics of the flux tube, such as the appearance of solitons with explosively growing amplitudes.

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