Gravitational descendants as generators of diffeomorphisms of the target space in topological Landau–Ginzburg gravity

A. Losev
Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia

I. Polyubin
L. D. Landau Institute for Theoretical Physics, RAS, 117334 Moscow, Russia

(Submitted 2 September 1993)
Pis'ma Zh. Eksp. Teor. Fiz. **58**, No. 8, 598–602 (25 October 1993)

Flows on the space of topological Landau–Ginzburg theories coupled to topological gravity have been studied. It is argued that flows corresponding to gravitational descendants change the target space from a complex plane to a punctured complex plane and lead to the motion of punctures. It is shown that the evolution of the topological theory due to these flows is given by dispersionless limit of the KP hierarchy.

1. Introduction. One of the most interesting features of topological matter coupled to topological gravity[1–6] is its connection to integrable systems. Specifically, the exponent of the generating function for correlators turns out to be the τ function of the integrable system. We argue that this connection can be explained as follows.

Observables can be identified with the tangent vectors to the space of topological theories. Each observable thus leads to the flow on the space of topological theories. The existence of the generating function for the correlators means that these flows commute.

Here we show that flows corresponding to descendants change in the target space of the theory: from a complex plane to a punctured complex plane. After that change the flows correspond to the motion of punctures. Integrable system which appears in the theory with one-dimensional target space with the superpotential X^p is a dispersionless limit of the p-reduced KP hierarchy.

2. Landau–Ginzburg topological matter theory on a punctured plane. The Landau–Ginzburg topological matter theory is a type-B, twisted, $N=2$ supersymmetric sigma model.[5,8] We consider as a target space of this model a punctured complex space with the coordinate X on it. The $N=2$ model is described by the Kahler prepotential, which we assume to be $|Q(X)|^2$, and by the superpotential W-polynomial of degree p. The Kahler metric $|Q'(X)|^2$ has zeros at the punctures. The Lagrangian of the model (written in terms of the twisted superfields) is:

$$L=\int d^4\theta|Q(X)|^2+\int d^2\theta_+W(X)+\int d^2\theta_-\bar{t}\bar{W}(\bar{X}). \qquad (1)$$

This model is well-defined for all values of $\bar{t}$, and correlators in topological matter are independent of $\bar{t}$, but in the coupling to topological gravity the $\bar{t}$ dependence appears (the holomorphic anomaly of Bershadsky, Cecotty, Ooguri, and Vafa). All our considerations below assume that $\bar{t}$ tends to zero.

The space of the local topological observables is given by the functions which are holomorphic everywhere except at the punctures. We take them in the form $P(X)/Q'(X)$, where P is a polynomial in X.

The n-point correlators in genus zero of the worldsheet in this topological matter theory are

$$\langle P_1/Q',\ldots,P_n/Q'\rangle^M_{W,Q}=\int_\Gamma \frac{P_1(X)/Q'\ldots P_n(X)/Q'(dQ)^2}{dW}. \tag{2}$$

Here the contour of integration Γ separates the zeros of dW from infinity and from the punctures. The superscript M indicates that this is an n-point correlator before the coupling to topological gravity. (The three-point correlator formula was introduced in Ref. 9.)

Note that $(dQ)^2$ in the numerator is due to the diffeomorphism anomaly in type-B topological theories.[10]

Equation (2) holds without any additional assumptions if deg $W' >$ deg Q'; otherwise, the infinite point of the complex plane gives a nonzero contribution. However, if we represent Q' as a perturbation of a constant and consider the n-point function as a series in perturbation theory, then *perturbatively* the infinite point does not contribute to the result. We will consider this subtle question in a separate publication.

3. Gravitational descendants. The gravitational descendants $\sigma_n(P/Q')$ can be constructed from matter fields using the same mechanism as in Ref. 11 (see also Ref. 12). They are

$$\sigma_n^{W,Q}(P/Q')=(dW/dQ)\int \sigma_{n-1}(P/Q')dQ. \tag{3}$$

Here $\sigma_0(\Phi)=\Phi$.

4. Multipoint correlators in topological theory coupled to topological gravity in genus zero of the worldsheet. These n-point correlators $\langle\Phi_1,\ldots,\Phi_n\rangle_{W,Q}$ are integrals over the moduli space of the complex structures of the worldsheet (the sphere with n marked points). For $n=3$ they are equal to the correlators in the topological matter itself (2) (since in this case the moduli space is given by a single point). For $n>3$ it is possible to write the recursion relations for these correlators and to express the n-point correlators in terms of $(n-1)$ point correlators. The recursion relations arise from the integration over the position of one of the marked points on the worldsheet. This integration results in an infinitesimal shift of the background plus contact terms. In our case the background consists of the pair W,Q; thus we expect that the fields will decompose in two sets: by shifting W and by shifting Q. Specifically, we uniquely decompose P/Q' as

$$P/Q'=S+RW'/Q', \tag{4}$$

where the degree of S is lower than the degree of W. According to Ref. 4, the S-term can then be interpreted as a shift of the superpotential, and the R-term, as we will see below, will correspond to the diffeomorphism which changes Q.

5. The S recursion relation. The S recursion relations have the form (for $n>2$)

$$\langle P_1/Q',\ldots, P_n/Q',S\rangle_{W,Q}=\frac{d}{dt}\langle P_1/Q',\ldots, P_n/Q'\rangle_{W+tS,Q}$$

$$+\sum_{i=1}^{n}\langle P_1/Q',\ldots,C_{W,Q}(P_i/Q',S),\ldots, P_n/Q'\rangle_{W,Q}. \qquad (5)$$

It can be argued[3,11,13] that the contact term is given by the representation

$$SP_1/Q'=\Phi/Q'+(W'/Q')\int C_{W,Q}(P_1/Q',S)dQ. \qquad (6)$$

This relation means that the product of the two fields (which move and stand at the marked point on the worldsheet) contains the first descendant of the contact term. The arbitrariness is in the choice of the field Φ. This arbitrariness can be fixed by self-consistency requirement. It turns out that in representation (6) Φ should have a degree lower than W. This contact term can be easily calculated:

$$C_{W,Q}(P_1/Q',S)=(1/Q')C_{W,X}(P_1,S)=(1/Q')(P_1S/W')'_+. \qquad (7)$$

Here the subscript stands for the positive part in the X expansion.

6. The R recursion relations which correspond to the integration over the position of the field RW'/Q' arise from the Ward identities which are connected with the diffeomorphism:

$$X\rightarrow X+tR(X)/Q'(X). \qquad (8)$$

Because of this diffeomorphism in the correlator $\langle P_1/Q',\ldots, P_n/Q'\rangle_{W,Q}$, its terms vary as follows:

$$W\rightarrow W+tW'R/Q',$$
$$Q\rightarrow Q+tR, \qquad (9)$$
$$P_i/Q'\rightarrow P_i/Q'+t(P_i/Q')'R/Q'=P_i/Q'-tC^{\mathrm{cl}}_{W,Q}(P_i/Q',RW'/Q').$$

The variation of superpotential in the action leads to the integral over the worldsheet which simulates the integral over the position of the insertion of $W'R/Q'$. The change of P_i due to this diffeomorphism is interpreted below as a minus in the "classical" contact term C^{cl}. We know, however, that there is also a "topological gravity" contact term C^{tg} [see Eq. (6)]:

$$P_iRW'/(Q')^2=(W'/Q')\int C^{\mathrm{tg}}_{W,Q}(P_i/Q',RW'/Q')dQ. \qquad (10)$$

Consequently,

$$C^{\mathrm{tg}}_{W,Q}(P_i/Q',RW'/Q')=(P_iR/Q')'/Q'. \qquad (11)$$

We are now ready to use the Ward identities; they are (for $n>2$)

$$\langle P_1/Q',..., P_n/Q',RW'/Q'\rangle_{W,Q}=\frac{d}{dt}(\langle P_1/Q',..., P_n/Q'\rangle_{W,Q-tR})|_{t=0}$$

$$+\sum_{i=1}^{n}\langle P_1/Q',...,C^{\rm tot}_{W,Q}(P_i/Q',RW'/Q'),..., P_n/Q'\rangle_{W,Q}, \tag{12}$$

where the total contact term is given by

$$C^{\rm tot}_{W,Q}(P_i/Q',RW'/Q')=C^{\rm tg}_{W,Q}(P_i/Q',RW'/Q')+C^{\rm cl}_{W,Q}(P_i/Q',RW'/Q')$$
$$=P_iR'/(Q')^2. \tag{13}$$

The R recursion relation thus takes the following simple form:

$$\langle P_1/Q',..., P_n/Q',RW'/Q'\rangle_{W,Q}=\frac{d}{dt}\langle P_1/(Q-tR)',..., P_n/(Q-tR)'\rangle_{W,Q-tR}|_{t=0}. \tag{14}$$

The R and S recursion relations, together with expression (2) of the three-point correlator, completely define the n-point correlators.

7. Integrable system. In topological theory coupled to topological gravity the central object is the generating function of the n-point correlators. Defining the correlators in the presence of a "formal exponent" as

$$\langle P_1,..., P_n;\exp(P)\rangle_W=\langle P_1,..., P_n\rangle_W+\langle P_1,..., P_n,P\rangle_W+\tfrac{1}{2}\langle P_1,..., P_n,P,P\rangle_W+...$$
$$+\tfrac{1}{k!}\langle P_1,..., P_n, P,P,..., P\rangle_W+..., \tag{15}$$

and using recursion relations (5) and (14), we can show that

$$\left\langle P_1/Q',..., P_n/Q';\exp\left(\sum_{k=1}^{\infty} t_kP_k/Q'\right)\right\rangle_{W,Q}=\langle P_1(t)/Q'(t),..., P_n(t)/Q'(t)\rangle_{W(t),Q(t)} \tag{16}$$

for some $W(t)$, $Q(t)$, and $P_i(t)$. In addition, if we define the polynomials $R_j(t)$ and $S_j(t)$ as the R and S parts of the polynomial $P_j(t)$,

$$P_j(X,t)=W'(X,t)R_j(t)+Q'(X,t)S_j(t), \tag{17}$$

then we can show that

$$\frac{\partial}{\partial t_j}P_i(t)=C_{W,X}[P_i(t),S_j(t)]=[P_i(t)S_j(t)/W'(t)]'_+, \tag{18}$$

$$\frac{\partial}{\partial t_j}W(t)=S_j(t), \tag{19}$$

$$\frac{\partial}{\partial t_j}Q(t)=-R_j(t). \tag{20}$$

8. Relation with dispersionless reductions of KP. Let $P_j(0)=X^j$. Using system (18)–(20), we can show that

A) The result of the evolution of P_j is equal to the derivative of the Hamiltonian of the dispersionless KP:

$$P_j(t)=(W(t)^{j/p})'_{+}. \quad (21)$$

B) The pair of functions (W,Q) evolve due to the dispersionless KP: If we introduce a Poisson bracket between the functions of t_1 and X

$$\{T_1(X,t_1),\, T_2(X,t_1)\}_1=\partial_1 T_1(X,t_1)\partial_X T_2(X,t_1)-\partial_1 T_2(X,t_1)\partial_X T_1(X,t_1), \quad (22)$$

then

$$\{W,Q\}_1=1 \quad (23)$$

and

$$\{W(X,t),\, [W(X,t)^{j/p}]_{+}\}_1=\partial_j W(X,t), \quad (24)$$

$$\{Q(X,t),\, [W(X,t)^{j/p}]_{+}\}_1=\partial_j Q(X,t). \quad (25)$$

C) The observable

$$D=\left\{\frac{p}{p+1}[W(t)^{p+1/p}]'_{+}-\sum_{j=1}^{\infty} t_j[W(t)^{j/p}]'_{+}\right\}/Q' \quad (26)$$

satisfies the dilaton equation

$$\langle D,P_1/Q',\dots,P_n/Q'\rangle_{W,Q}=(n-2)\langle P_1/Q',\dots,P_n/Q'\rangle_{W,Q}. \quad (27)$$

Comparing this object with the dilaton which is a descendant of a puncture, we obtain the Krichever form of the string equation in genus zero[9]

$$\frac{p}{p+1}[W(t)^{p+1/p}]'_{+}-\sum_{j=1}^{\infty} t_j[W(t)^{j/p}]'_{+}=QW'. \quad (28)$$

This equation algebraically determines W and Q in terms of the times t.

D) Finally, $\langle ;\exp(\Sigma_k k t_k X^{k-1})\rangle_{W,X}$ is a logarithm of the τ function of the dispersionless W-reduced KP hierarchy.

9. Conclusions. The approach used in this letter should be generalized to higher dimensions of the target space and to different topologies of the target space (for some steps in this direction see Refs. 9 and 14). From the physical point of view, it should be generalized to the higher genera of the worldsheet (in A_n singularities, see Ref. 15). Development in this direction seems to link such different theories as singularity theory, geometry of moduli space of Riemann surfaces, and two-dimensional quantum gravity.

We are deeply indebted to R. Dijkgraaf, A. Gerasimov, V. Fock, I. Krichever, S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, N. Nekrasov, A. Rosly, K. Saito, and A. Varchenko for fruitful discussions.

[1] E. Witten, Nucl. Phys. B **340**, 281 (1990).
[2] R. Dijkgraaf and E. Witten, Nucl. Phys. B **342**, 486 (1991).
[3] E. Verlinde and H. Verlinde, Nucl. Phys. B **348**, 457 (1991).

[4] R. Dijkgraaf, E. Verlinde, and H. Verlinde, Proceedings of the Trieste Spring School, 1990; M. Green *et al.*, eds. (World Scientific, 1991).
[5] K. Li, Nucl. Phys. B **354**, 711 (1991).
[6] M. Kontsevich, Comm. Math. Phys. **147**, 1 (1992).
[7] C. Vafa, Mod. Phys. Lett. A **6**, 337 (1990).
[8] T. Eguchi and S. K. Yang, Mod. Phys. Lett. A **5**, 1663 (1990).
[9] I. M. Krichever, Preprint LPTENS-92/18 (hepth 9205110).
[10] E. Witten, *Mirror Manifolds and Topological Field Theory*, in S.-T. Yau, ed., Essays On Mirror Manifolds (International Press, 1992).
[11] A. Losev, *Descendants Constructed from Matter Fields*; K. Saito, *Higher Residue Pairing*, in *Topological L–G Theories Coupled To Topological Gravity*, Preprint TPI-MINN-40T (May, 1992); A. Losev, TMP (Teoreticheskaya i matematicheskaya Fizika, in English) **95**, No. 2, 307 (1993).
[12] T. Eguchi *et al.*, *Topological Strings, Flat Coordinates, and Gravitational Descendants*, University of Tokyo, Preprint UT 630, February, 1993.
[13] R. Dijkgraaf, Talk on RIMS Conference "Infinite dimensional analysis," July, 1993.
[14] B. Dubrovin, Nucl. Phys. B **379**, 627 (1992).
[15] S. Kharchev *et al.*, Nucl. Phys. B **380**, 181 (1992).

Published in English in the original Russian journal. Reproduced here with the stylistic changes by the Translations Editor.