Higher Hamiltonian structures (the sl_2 case)

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The procedure of Magri and Weinstein is applied to produce an infinity of compatible Poisson structures on a bihamiltonian manifold, to the case of the KdV phase space. The higher Gel'fand–Dikii structures thus obtained contain nonlocal terms which are expressed with the help of the r.h.s. of the KdV hierarchy. A generating function for all these Poisson structures is given in terms of the Baker–Akhiezer functions. The symplectic leaves of these Poisson structures are described.

1. DEFINITION OF THE HIGHER HAMILTONIAN STRUCTURES

In [M], [W] it is explained how to associate to a pair of Poisson structures V_i, $i=1,2$, on a manifold P, one of which (V_1) is symplectic, an infinite sequence V_n of compatible Poisson structures. The maps V_i: $T^*P\to TP$ allow us to define the Nijenhuis operator Λ: $TP\to TP$ by $V_2V_1^{-1}$; we then pose $V_n=\Lambda^{n-1}V_1$.

A natural example of compatible Poisson structures is provided by the first Gel'fand–Dikii and second (Adler–)Gel'fand–Dikii (GD1 and GD2) structures on manifolds of differential operators. Note that GD1 is not exactly symplectic, since if we define the manifold to be $\mathscr{L}_2=\{\partial^2+u,\ u\in C_c^\infty(\mathbf{R})\}$, then $T_L\mathscr{L}_2=\{\delta u,\ \delta u\in C_c^\infty(\mathbf{R})\}$ (the subscript c means compactly supported functions) for $L\in\mathscr{L}_2$, and $T_L^*\mathscr{L}_2$ is the set of forms given by $\delta u\to\int_{\mathbf{R}}\xi\delta u$, $\xi\in C_c^\infty(\mathbf{R})$. The map V_1 is then $T_L^*\mathscr{L}_2\to T_L\mathscr{L}_2$, $\xi\mapsto\xi'$; it has a cokernel of dimension one.

A natural way to overcome this difficulty is to enlarge the space of functionals on $\mathscr{L}_2$ (recall that in the usual formalism it is the set of maps $u\mapsto\int_{\mathbf{R}}f[x,u(x),...,u^{(k)}(x)]\,dx$, f smooth with compact support in x and polynomial in the other variables).

We now pose Fun $\mathscr{L}_2$ to be the linear span of the functionals

$$u\mapsto\int_{\mathbf{R}^k}f_1[x_1,u^{(\alpha)}(x_1)]...f_k[x_1,u^{(\alpha)}(x_k)]\prod_{\alpha=1}^{N}\varepsilon(x_{i_\alpha}-x_{j_\alpha})\prod_{i=1}^{k}dx_i,\quad i_\alpha<j_\alpha,$$

where $\cup_{\alpha=1}^{k-1}\{j_\alpha,j_\alpha\}=\{1,...,k\}$, f_k is a polynomial in the $u^{(\alpha)}(x_k)$, $\alpha\geqslant0$ and the coefficients C_c^∞ are functions of x_k; ε is the Heaviside function [$\varepsilon(x)=-\frac{1}{2}$ if $x<0$; otherwise, it is $\frac{1}{2}$]; and N is arbitrary.

We still set $T_L\mathscr{L}_2=\{\delta u,\ \delta u\in C_c^\infty(\mathbf{R})\}$, but define now $T_L^*\mathscr{L}_2$ as the set of forms given by $\delta u \mapsto \int_{\mathrm{R}} \tilde{\xi} \delta u$, $\tilde{\xi}$ is a smooth function on R with opposite limits at $+\infty$ and $-\infty$. The map V_1: $T_L^* \mathscr{L}_2 \to T_L \mathscr{L}_2$, $\tilde{\xi} \mapsto \tilde{\xi}'$ is now a linear isomorphism, and we can apply the Magri–Weinstein procedure.

Let us consider in $T_L^* \mathscr{L}_2$ the subspace T_0^* which is linearly spanned by the functions

$$x_1 \mapsto \int_{\mathrm{R}^{k-1}} f_1[x_1, u^{(\alpha)}(x_1)] \dots f_k[x_k, u^{(\alpha)}(x_k)] \prod_{\alpha=1}^{N} \varepsilon(x_{i_\alpha} - x_{j_\alpha}) \prod_{i=2}^{k} dx_i,$$

where exactly one i_α or j_α is equal to 1 and $f_1, \dots, f_k$ are as above, but the constant term of f_1 is similar to the constant $+C_c^\infty$ (N is arbitrary); and in $T_L \mathscr{L}_2$ the subspace T_0 is linearly spanned by analogous expressions, with the conditions that the constant terms of the polynomial f_i are C_c^∞ (and the other terms are smooth), and there is no restriction on (i_α), (j_α).

The map V_1 induces a linear isomorphism between T_0^* and T_0: Indeed, $V_1(T_0^*) \subset T_0$, and the preimage of the element of T_0

$$x_1 \mapsto \int_{\mathrm{R}^{k-1}} f_1[x_1, u^{(\alpha)}(x_1)] \dots f_k[x_k, u^{(\alpha)}(x_k)] \prod_{\alpha=1}^{N} \varepsilon(x_{i_\alpha} - x_{j_\alpha}) \prod_{i=2}^{k} dx_i$$

is

$$x_0 \mapsto \int_{\mathrm{R}^k} 1(x_0) f_1[x_1, u^{(\alpha)}(x_1)] \dots f_k[x_k, u^{(\alpha)}(x_k)] \cdot \varepsilon(x - x_1)$$

$$- \prod_{\alpha=1}^{N} \varepsilon(x_{i_\alpha} - x_{j_\alpha}) \prod_{i=1}^{k} dx_i,$$

which belongs to T_0^* [$1(x_0)$ is a constant function of x_0 equal to 1]. On the other hand, $V_2(T_0^*) \subset T_0$ [recall that $V_2 = \frac{1}{4}(\partial^3 + 4u\partial + 2u_x)$]. Note also that V_1 and V_2 define antisymmetric bilinear forms on T_0^* [or, more generally, on the space $T^*(-1)$, if $T^*(c) = \{\xi \in C^\infty(\mathrm{R}) \mid \xi$ is constant at both infinities and $\lim_{+\infty} \xi = c \lim_{-\infty} \xi\}$; if $\xi, \eta \in T^*(c)$, $\langle V_1(\xi), \eta \rangle + \langle \xi, V_1(\eta) \rangle = (1 - c^2) \lim_{+\infty} \xi \lim_{+\infty} \eta$; and V_1 is not bijective from $T^*(1)$ to $C_c^\infty(\mathrm{R})$. On the other hand, V_2 defines an antisymmetric form on any $T^*(c)$]. We can verify that V_1 and V_2 still define the Poisson structures on Fun $\mathscr{L}_2$. The recursion operator Λ is then well defined from T_0 to itself; we obtain mappings $V_n = \Lambda^n \partial$ from T_0^* to T_0.

If now F belongs to Fun $\mathscr{L}_2$, dF belongs to T_0^*, and $V_n(dF)$ belongs to T_0, then let G be another function of Fun $\mathscr{L}_2$. It is easy to see that $V_n(dF)G$ still belongs to Fun $\mathscr{L}_2$. Thus, the V_n define bilinear operations on Fun $\mathscr{L}_n$. The Magri–Weinstein arguments allow us to show that they form an infinite family of compatible Poisson brackets which we call higher Gel'fand–Dikii brackets. We will now analyze these brackets more closely.

2. NONLOCAL PART OF THE HIGHER GEL'FAND-DIKII STRUCTURES

Recall that on the KdV phase space $\mathscr{L}_2=\{\partial^2+u,\ u\in C_c^\infty(R)\}$, *GD*1 *is given by the operator* $V_1=\partial$ *and GD*2 *is given by* $V_2=\frac{1}{4}(\partial^3+2u\partial+2u_x)$, *which leads to the brackets*

$$\{u(x),u(y)\}_1=\delta'(x-y),\quad \{u(x),u(y)\}_2=\frac{1}{4}\delta'''(x-y)+\frac{1}{2}[u(x)+u(y)]\delta'(x-y).$$

The recursion operator is $\Lambda=\frac{1}{4}(\partial^2+4u+2u_x\partial^{-1})$. The operator of the nth GD structure will then be $\Lambda^n\partial$. The result of this section is that $\Lambda^n\partial$ can be written (local part) $+\Sigma_{i=1}^n K_i'\partial^{-1}K_{n-i}\partial$, where the local part is a differential operator (where the coefficients are differential polynomials in u), and where K_n are the right-hand sides of the KdV hierarchy: $K_{n+1}=\Lambda^*K_n$ [we set $(f\partial^k)^*=(-\partial)^kf$; f is a function and k is an integer), $K_0=1$; K_n is then a polynomial differential in u, which contains (by assumption) no constant term if $n\neq 0$.

We first prove several relations in the algebra of formal pseudodifferential operators:

Lemma. If a, b, a_1, and b_1 are differential polynomials in u, and P is a differential operator, we have

$$P(a\partial^{-1}b)=[Pa]\partial^{-1}b+R,$$

$$(a\partial^{-1}b)P=a\partial^{-1}[P^*b]+S,$$

$$(a\partial^{-1}b)(a_1\partial^{-1}b_1)=[a\partial^{-1}ba_1]\partial^{-1}b_1+a\partial^{-1}[(a_1\partial^{-1}b_1)^*b],$$

where, e.g., $[Pa]$ denotes the function obtained from the operation of P on a, and R and S are differential operators (in the last relation ba_1 is assumed to be a total derivative c', and $[\partial^{-1}ba_1]=c$ in the r.h.s.).

The first relation is proved by examining the cases $P=$function and $P=\partial$, and by the remark that the set of P satisfying it forms an algebra; the second relation can be deduced from it by applying $*$. To obtain the third one, we replace in the r.h.s. ba_1 by $\partial[\partial^{-1}ba_1]-[\partial^{-1}ba_1]\partial$.

We apply this lemma to the computation of the nonlocal part of Λ^n. $\Lambda=$(local part) $+K_1'\partial^{-1}$, since $K_1=u/2$. Assume that Λ^n is written $P_n+\Sigma_{i=1}^n K_i'\partial^{-1}K_{n-i}$, P_n is a differential operator. Then

$$\Lambda^{n+1}=\text{differential operator}+\frac{1}{4}[P_n.2u_x]\partial^{-1}+\sum_{i=1}^n K_i'\partial^{-1}\left[\frac{1}{4}(\partial^2+4u)^*K_{n-i}\right]$$
$$+\sum_{i=1}^n K_i'\left[\frac{\partial^{-1}}{4}K_{n-i}2u_x\right]\partial^{-1}-\frac{1}{4}\sum_{i=1}^n K_i'\partial^{-1}[\partial^{-1}2u_xK_{n-i}]$$
$$=\left[\Lambda^n\frac{u_x}{2}\right]\partial^{-1}+\sum_{i=1}^n K_i'\partial^{-1}K_{n+1-i}+\text{diff. op.}$$

$$=K'_{n+1}\partial^{-1}+\sum_{i=1}^{n}K'_i\partial^{-1}K_{n+1-i}+\text{diff. op.},$$

since from $\Lambda^*=\partial^{-1}\Lambda\partial$ it follows that $K'_{n+1}=\Lambda K'_n$, and so $K'_{n+1}=\Lambda^nK'_1=\Lambda^n(u_x/2)$. From $K_{s+1}=\Lambda^*K_s$ it follows that $2u_xK_s$ is the total derivative of a differential polynomial in u; in $[\partial^{-1}2u_xK_s]$ we set the constant term of this polynomial to zero.

This establishes $\Lambda^n=\text{local part}+\Sigma_{i=1}^{n}K'_i\partial^{-1}K_{n-i}$ by induction. Our result follows. The Poisson bracket for GDn is then

$$\{u(x),u(y)\}_n=(\text{local part})+\sum_{i=1}^{n-1}K'_i(x)\partial_x^{-1}K_{n-1-i}(x)\partial_x\delta(x-y)$$

$$=(\text{local part})-\sum_{i=1}^{n-2}K'_i(x)K'_{n-1-i}(y)\varepsilon(x-y),$$

$$(\text{local part})=\frac{1}{4^{n-1}}\delta^{(2n-1)}(x-y)+\sum_{i=0}^{2n-2}p_i(u)(x)\delta^i(x-y),$$

where the p_i are the differential polynomials in u.

3. A GENERATING FUNCTION FOR THE HIGHER GD STRUCTURES

Let ψ_λ and ψ^*_λ be the conjugated wave (Baker–Akhiezer) functions for ∂^2+u. In other words, $(\partial^2+u)\psi_\lambda=\lambda^2\psi_\lambda$, $(\partial^2+u)\psi^*_\lambda=\lambda^2\psi^*_\lambda$, $\psi_\lambda(x)=e^{\lambda x}[1+\Sigma_{i\geqslant 1}u_i(x)\lambda^{-i}]$, $\psi^*_\lambda(x)=e^{-\lambda x}[1+\Sigma_{i\geqslant 1}u_i(x)(-\lambda)^{-i}]$, $u_i(x)=0$ for x large negative enough. We set $R_+=\psi_\lambda^2$, $R_0=\psi_\lambda\psi^*_\lambda$, and $R_-=\psi^{*2}_\lambda$. The R_i satisfy $\frac{1}{4}(\partial^3+4u\partial+2u_x)R_i=\lambda^2\partial R_i$. We then have the identity

$$\frac{\lambda^2}{\Lambda-\lambda^2}=\frac{1}{2}R'_+\partial^{-1}R_-+\frac{1}{2}R'_-\partial^{-1}R_+-R'_0\partial^{-1}R_0. \tag{1}$$

It follows from

$$\frac{1}{4}[\partial^2+4(u-\lambda^2)+2u_x\partial^{-1}]\frac{1}{2}R'_+\partial^{-1}R_-=\frac{1}{2}R'_+\left(\frac{1}{4}\partial^2+u-\lambda^2\right)\partial^{-1}R_-$$

$$+\frac{1}{8}(2R''_++R'''_+)\partial^{-1}R_-+\frac{u_x}{4}R_+\partial^{-1}R_--\frac{u_x}{4}\partial^{-1}R_+R_-$$

$$=\frac{1}{8}\partial R'_+R_-+\frac{1}{4}R''_+R_--\frac{u_x}{4}\partial^{-1}R_+R_-;$$

after summation of the two other terms, the last expression gives zero. We then have

$$(\Lambda-\lambda^2)\left(\frac{1}{2}R'_+\partial^{-1}R_-+\frac{1}{2}R'_-\partial^{-1}R_+-R'_0\partial^{-1}R_0\right)$$

$$=\frac{1}{8}(R''_+R_-+R''_-R_+-2R''_0R_0)=\frac{1}{4}(R'^2_0-R'_+R'_-)$$

$$=\frac{1}{4}(\psi_\lambda\psi_\lambda^{*\prime}-\psi_\lambda'\psi_\lambda^*)^2=\lambda^2$$

(the Wronskian of ψ_λ and ψ_λ^* is constant, and it takes the value 2λ at $-\infty$).

From (1) follows a formula that generates all the $\Lambda^k\partial$:

$$-\sum_{k\geqslant 0}\frac{\Lambda^k\partial}{\lambda^{2k}}=-\frac{1}{2}R_+'\partial^{-1}R_-'-\frac{1}{2}R_-'\partial^{-1}R_+'+R_0'\partial^{-1}R_0'$$

(we have used $R_+R_-=R_0^2$). Setting $R_\pm=e^{\pm 2\lambda x}\hat{R}_\pm$, we have $R_\pm'=e^{\pm 2\lambda x}(\hat{R}_\pm'\pm 2\lambda\hat{R}_\pm)$, and so

$$-\sum_{k\geqslant 0}\frac{\Lambda^k\partial}{\lambda^{2k}}=-\frac{1}{2}(\hat{R}_+'+2\lambda\hat{R}_+)(\partial-2\lambda)^{-1}(\hat{R}_-'-2\lambda\hat{R}_-)-\frac{1}{2}(\hat{R}_-'-2\lambda\hat{R}_-)$$
$$\times(\partial+2\lambda)^{-1}(\hat{R}_+'+2\lambda\hat{R}_+)+R_0'\partial^{-1}R_0'.$$

Here $(\partial\pm 2\lambda)^{-1}$ should be expanded as $\Sigma_{k\geqslant 0}\partial^k/(\pm 2\lambda)^{k+1}$. The only nonlocal contribution is therefore one of the terms in R_0, which enables us to recover the results of the last section (since $R_0=1+\Sigma_{n\geqslant 1}K_n/\lambda^{2n}$). We note that all the expressions contained in the expansion in λ are polynomial differentials in u since they are invariant under the transformations $\psi_\lambda(x)\mapsto c(\lambda)\psi_\lambda(x)$, $\psi_\lambda^*(x)\mapsto c(\lambda)^{-1}\psi_\lambda^*(x)$, $c(\lambda)\in\mathbb{C}[[\lambda^{-1}]]^*$.

If w is then a primitive of the variable u, we obtain

$$\{w(x),w(y)\}_\lambda=\sum_{n\geqslant 0}\frac{\{w(x),w(y)\}_n}{\lambda^{2n}}$$
$$=\left(\frac{1}{2}R_+(x)R_-(y)+\frac{1}{2}R_-(x)R_+(y)-R_0(x)R_0(y)\right)\varepsilon(x-y).$$

It should be noted that since

$$R_+(x)R_-(y)+R_-(x)R_+(y)-2R_0(x)R_0(y)=[\psi_\lambda(x)\psi_\lambda^*(y)-\psi_\lambda^*(x)\psi_\lambda(y)]^2,$$

$$\{w(x),w(y)\}_\lambda=[\psi_\lambda(x)\psi_\lambda^*(y)-\psi_\lambda^*(x)\psi_\lambda(y)]^2\varepsilon(x-y).$$

Here $e^{\pm 2\lambda(x-y)}\varepsilon(x-y)$ should be expanded as $-\Sigma_{k\geqslant 0}\delta^{(k)}(x-y)/(\pm 2\lambda)^{k+1}$.

In the case of GD3 the bracket is

$$\{u(x),u(y)\}_3=\frac{1}{16}\delta^V(x-y)+\frac{1}{4}[u(x)+u(y)]\delta'''(x-y)+\frac{1}{8}[u'(x)$$
$$-u'(y)]\delta''(x-y)+\frac{1}{2}[u(x)^2+u(y)^2]\delta'(x-y)$$
$$-\frac{1}{4}u'(x)u'(y)\varepsilon(x-y).$$

For this third GD structure the functional $\int_{\mathbf{R}} u(x)dx$ is a Hamiltonian for the KdV equation. It is not clear what the Hamiltonians for the KdV equations are in the higher GD structures [they should be the nonlocal quantities which are related to the solutions of $(\partial^2+u+\lambda)\varphi=0$ for small λ].

Remarks. 1. R_+ and R_- can also be used to give a generating function for "A operators" for Λ:

$$\frac{\partial\Lambda}{\partial t}=\sum_{n\geqslant 0}\frac{1}{\lambda^n}\frac{\partial\Lambda}{\partial t_n}=[\Lambda,R'_+\partial^{-1}R_- - R'_-\partial^{-1}R_+],$$

where t_i are the times of the KdV hierarchy.

2. In the last section we used the fact that $u_x K_n$ is a total derivative. We can show more generally that for any i and j, $K'_i K_j$ is a total derivative. Recalling that $R_0=1+\Sigma_{n\geqslant 1}K_n/\lambda^{2n}$, it is sufficient to prove that $R_0(\lambda,x)R'_0(\mu,x)$ is a total derivative. Indeed, [noting that $W(f,g)=fg'-f'g$]

$$[W(\psi_\lambda,\psi^*_\mu)W(\psi^*_\lambda)]'=(\mu^2-\lambda^2)[R_0(\lambda)R'_0(\mu)-R'_0(\lambda)R_0(\mu)],$$

so that a primitive of $R_0(\lambda)R'_0(\mu)$ is

$$\frac{1}{2}\left(\frac{1}{\mu^2-\lambda^2}W(\psi_\lambda,\psi^*_\mu)W(\psi^*_\lambda,\psi_\mu)+R_0(\lambda)R_0(\mu)\right);$$

it consists of differential polynomials in u, since it is invariant under the transformations $\psi_\lambda(x)\mapsto c(\lambda)\psi_\lambda(x)$, $\psi^*_\lambda(x)\mapsto c(\lambda)^{-1}\psi^*_\lambda(x)$, $\psi_\mu(x)\mapsto d(\lambda)\psi_\mu(x)$, $\psi^*_\mu(x)\mapsto d(\lambda)^{-1}\psi^*_\mu(x)$, $c(\lambda)\in\mathbf{C}[[\lambda^{-1}]]^*$, $d(\mu)\in\mathbf{C}[[\mu^{-1}]]^*$.

4. SYMPLECTIC LEAVES OF THE HIGHER GD STRUCTURES

Let M be a finite-dimensional manifold which is endowed with a symplectic structure V_0: $T^*M\rightarrow TM$ and a Poisson structure V_1: $T^*M\rightarrow TM$, which we assume to be compatible. Let $\Lambda=V_1V_0^{-1}$ be the recursion operator. Let $P(\Lambda)=\Pi_{i=1}^n(\Lambda-\lambda_i)^{k_i}$ be an arbitrary monic polynomial (the λ_i are all different, $n\geqslant 1$). The general form of the Poisson structures defined in [W] will then be $V_P=P(\Lambda)V_0$.

Let us analyze the symplectic leaves of V_P. They are the integral manifolds of the forms ξ such that $P(\Lambda)V_0\xi=\Pi_{i=1}^n(\Lambda-\lambda_i)^{k_i}V_0\xi=0$. Any ξ satisfying this relation is such that at any point x of M, $\xi(x)$ belongs to the sum $\Sigma_{i=1}^n D_i(x)$ of the subspaces $D_i(x)$ of T^*_xM which consist of the forms satisfying $(\Lambda-\lambda_i)^{k_i}V_0\xi=0$. The tangent vectors to the symplectic leaves of V_P therefore are exactly the vectors tangent to the symplectic leaves of all the $V_{(\Lambda-\lambda_i)^{k_i}}$, and the symplectic leaves of V_P are the intersections of the symplectic leaves of the $V_{(\Lambda-\lambda_i)^{k_i}}$.

Let us apply this discussion to the structures on the KdV phase space discussed above. First, we describe the symplectic leaves of $V_{\lambda^k V_0}(k\geqslant 1)$. For this purpose, we solve $\Lambda^k V_0\xi=0$, where ξ is a covector at ∂^2+u. Recall ([K]) that the Casimir functions of V_1 are the functions of Tr $M(u)$ [$M(u)$ is the monodromy operator of ∂^2+u], so that $\Lambda V_0\xi=0$ means that ξ is proportional to d Tr $M(u)$. Let us solve $\Lambda^2V_0\xi=0$. Differentiating $(\Lambda+\lambda)V_0 d$ Tr $M(u+\lambda)=0$, we obtain $V_0 d$ Tr $M(u)+\Lambda V_0 d\partial_\lambda$ $\times$Tr $M(u+\lambda)|_{\lambda=0}=0$. $\Lambda^2V_0\xi=0$ means that $\Lambda V_0\xi$ is proportional to d Tr $M(u)$, i.e., $V_0\xi$ is a linear combination of $d\,\mathrm{Tr}\,M(u+\lambda)$ and $d\partial_\lambda\,\mathrm{Tr}\,M(u+\lambda)|_{\lambda=0}$. In the same way, we see that $\Lambda^k V_0\xi=0$ has for solutions the linear combinations of $d\,\mathrm{Tr}\,M(u)$, $d\partial_\lambda\,\mathrm{Tr}\,M(u+\lambda)|_{\lambda=0},\ldots,d\partial_{\lambda^{k-1}}\,\mathrm{Tr}\,M(u+\lambda)|_{\lambda=0}$. Hence the symplectic leaves of $\Lambda^k V_0$ are the manifolds

$$\mathrm{Tr}M(u)=C_0,\ldots,\partial_\lambda^{k-1}\mathrm{Tr}M(u+\lambda)|_{\lambda=0}=C_{k-1},$$

where $C_0,\ldots,C_{k-1}$ are constants. In the Miura coordinates $[\partial^2+u=(\partial+\varphi')(\partial-\varphi')$, $\varphi(-\infty)=0]$ these conditions can be written

$$(e^{\varphi}+e^{-\varphi})(+\infty)=C_0,$$

$$e^{\varphi(\infty)}\int_{-\infty}^{\infty}dxe^{-2\varphi(x)}\int_{-\infty}^{x}e^{2\varphi}+e^{-\varphi(\infty)}\int_{-\infty}^{\infty}dxe^{2\varphi(x)}\int_{-\infty}^{x}e^{-2\varphi}=C_1,$$

$$e^{\varphi(\infty)}\int_{-\infty}^{\infty}dxe^{-2\varphi(x)}\int_{\infty}^{x}dye^{2\varphi(y)}\int_{-\infty}^{y}dze^{-2\varphi(z)}\int_{-\infty}^{z}e^{2\varphi}$$

$$+e^{-\varphi(\infty)}\int_{-\infty}^{\infty}dxe^{2\varphi(x)}\int_{-\infty}^{x}dye^{-2\varphi(y)}\int_{-\infty}^{y}dze^{2\varphi(z)}\int_{-\infty}^{z}e^{-2\varphi}=C_2,\ldots .$$

Remark. It would be of interest to compute the Poisson brackets between the monodromy operators of the $\partial^2+u+\lambda$, where the Poisson brackets of u correspond to the higher GD structures. In the case of GD1 the Poisson brackets between monodromies are given by the rational r-matrix $t/(\lambda-\mu)$ on the group $SL_2(\mathrm{C}[[\lambda]])$ (Faddeev–Takhtajan). In the case of GD2 they are given by the trigonometric r-matrix $\frac{1}{2}(\lambda+\mu)/(\lambda-\mu)t+r$ (in the Belavin–Drinfeld notation). These two Poisson structures on $SL_2(\mathrm{C}[[\lambda]])$ are compatible.

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