Solution of the Korteweg–de Vries equation which arises near the breaking point in problems with a slight dispersion

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A special solution of the Korteweg–de Vries equation describing the beginning of "breaking" in problems with a slight dispersion is analyzed. It is suggested that this solution simultaneously satisfies an ordinary differential equation which is the steady-state part of the symmetry of the Korteweg–de Vries equation. Arguments in favor of this suggestion are presented.

1. Gurevich and Pitaevskiĭ[1,2] have shown that near the point corresponding to the beginning of the "wavefront" breaking in problems with a slight dispersion it is necessary to use a special solution of the Korteweg–de Vries equation

$$v_t+v_{xxx}+vv_x=0. \tag{1}$$

Outside the region of fast oscillations,[1,2] the asymptotic form of the solution of this equation is determined by the joining condition

$$v(t,x)\approx|t|^{1/2}f(s)(|t|\gg 1), \tag{2}$$

where $s=x/|t|^{3/2}$, and $f(s)$ is the unique root of the equation

$$s-(\mathrm{sgnt})f+f^3=0. \tag{3}$$

The interest in the properties of the solution of (1) has not waned.[3–8] In the present letter we suggest (and present arguments to support this suggestion) that this solution simultaneously satisfies the ordinary differential equation

$$v_7+7[vv_5+3v_1v_4+5v_2v_3]/3+35[v_3v^2+v_1^3+4v_2v_1v]/18+35v_1v^3/54$$
$$+5[xv_1-3t(v_3+vv_1)+2v]/54=0, \tag{4}$$

which is the steady-state part of the symmetry of the Korteweg–de Vries equation[9] (the subscript specifies the order of the derivative of v with respect to x). It follows from Eqs. (1) and (4) that v must also be a solution of an ordinary differential equation in the variable t.

2. The suggestion that the Gurevich–Pitaevskiĭ solution of (1) is simultaneously an exact solution of (4) is based in particular on the form of its complete (to an arbitrary power of $|t|^{-1}$) asymptotic expansion outside the oscillation zone. However, the form of the complete expansion has not been regarded as important. At this point it must be taken up.

It was pointed out especially in Ref. 2 that this Korteweg–de Vries solution is of general importance: it "always" arises near a breaking point, provided that dissipative effects are not important. In turn, to study the properties of the Gurevich–Pitaevskiĭ solution, one can use any of the original problems with a slight dispersion. An initial-value problem has previously been used frequently for this purpose for the Korteweg–de Vries equation. Despite its rigor, that approach slightly "obscures" the universal aspect of the Gurevich–Pitaevskiĭ solution. To avoid this problem, we will work from a more general formulation to refine the asymptotic result in (2).

3. Il'in[10] has carried out an exhaustive analysis of the breaking of a simple wave for an equation with a slight dissipation, $U_\tau+[\varphi(U)]'_z=\epsilon^2U_{zz}$. We replace the second derivative by a third derivative,

$$U_\tau+[\varphi(U)]'_z+\epsilon^2U_{zzz}=0, \tag{5}$$

and we specify a monotonic initial profile for U,

$$U|_{\tau=-1}=w(z). \tag{6}$$

The question of the asymptotic behavior as $\epsilon\to 0$ of the solution of initial-value problem (5), (6) near the breaking point will be studied for the case of an extremely general nonlinearity of φ –essentially the same as in Ref. 10. Specifically, we assume that the following conditions, which guarantee a breaking at the point $z=0$, $\tau=0$ $[g(\alpha)=\varphi'(w(\alpha))]$, hold:

$$\varphi(0)=\varphi'(0)=\omega(0)=0,\quad g(0)=g''(0)=0,\quad g'(0)=-1,\quad g'''(0)=6,\quad \varphi''(0)=1. \tag{7}$$

(We may think of the standard particular case $\varphi=U^2/2$ of the Korteweg–de Vries equation and, for example, an initial profile $\omega=-3^{-1/2}\arctan(3^{1/2}z)$, but no particular simplification is achieved as a result.)

4. The solution of (5), (6) before the "wave breaking" is sought in the form of a series

$$U=U_0+\epsilon U_1+\ldots+\epsilon^k U_k+\ldots, \tag{8}$$

which is a so-called outer expansion.[10] The quantity $U_0(\tau,z)$ is the solution of the limiting ($\epsilon=0$) problem for (5), (6). On the characteristics

$$z=\alpha+g(\alpha)(1+\tau) \tag{9}$$

of the limiting equation, this function is constant (under the condition $z_\alpha'>0$):

$$U_0(\tau,z)=\omega(\alpha). \tag{10}$$

For other $U_k(\tau,z)$, one writes a recurrent series of linear equations which are integrable in quadrature.

Proceeding with the same arguments by induction, as in Ref. 10, we can easily show that in the limits $z\to 0$ and $\tau\to 0$ ($k\geqslant 1$) we have

$$U_k(\tau,z)=\Sigma[h(\tau,\alpha)]^{-i}\alpha^j\Phi_{ij}(\alpha). \tag{11}$$

Here $h(\tau,\alpha)=z'_\alpha=1+g'(\alpha)(\tau+1)$; $\Phi_{ij}(\alpha)$ are smooth functions; z, τ, and α are related by (9); and the final sum is over integers $i, j\geqslant 0$, for which the conditions $4\leqslant 2i\leqslant 7k-j-1$ hold. The singularities of the functions U_k at the breaking point thus increase with increasing k. Outer expansion (8) becomes inapplicable here.

To find the correct approximation of $U(\epsilon,\tau,z)$ near the origin, we introduce the extended variables $\tau=\epsilon^{4/7}t$, $z=\epsilon^{6/7}x$, and we seek an inner asymptotic expansion in the form of the series

$$U(\epsilon,\tau,z)\equiv W(\epsilon,t,x)=\epsilon^{2/7}(W_0+\epsilon^{2/7}W_1+...+\epsilon^{2n/7}W_n+...). \tag{12}$$

Here, as in Ref. 10, the change of variables is guided by dimensionality considerations concerning Eq. (5) and relation (9). The form of the inner expansion in (12) is dictated by the condition that it be compatible with (8). Equations for W_k are found in the usual way after (12) is substituted into the original equation, (5). We find that $W_0(t,x)$ is a solution of the Korteweg–de Vries equation (1). In accordance with the discussion above, it is also the basic subject of study of the present paper: the special Gurevich–Pitaevskiĭ solution.

If we replace the functions $U_k(\tau,z)$ by their asymptotic expressions at the origin in (8), using representations (11), relation (9), and a change of independent variable, we find a series of the type in (12) in which the functions $W_n(t,x)$ are replaced by their asymptotic series. In particular, for $n=0$ this procedure provides the answer to the question which is actually the reason why we are taking up problem (5), (6) in the present letter. The complete asymptotic expansion of the Gurevich–Pitaevskiĭ solution of the Korteweg–de Vries equation outside the oscillation region is the series

$$v=W_0=|t|^{1/2}\left[f(s)+\sum_{j=1}^{\infty}|t|^{-7j/2}g_j(s)\right](|t|\gg 1), \tag{13}$$

where f is the unique solution of cubic equation (3), and $g_i(s)$ are specific continuous functions.

5. It is a simple matter to verify directly that a certain joint asymptotic solution $\Gamma(t,x)$ of Korteweg–de Vries equation (1) and the ordinary differential equation in (4) has a form similar to (13) at $|t|\gg 1$:

$$\Gamma(t,x)=|t|^{1/2}\left[f(s)+\sum_{j=1}^{\infty}|t|^{-7j/2}p_j(s)\right]. \tag{14}$$

The coefficients $p_j(s)$ here are determined unambiguously from algebraic recurrence relations of the function. These functions are continuous in the region in which the root $f(s)$ of Eq. (3) is single-valued.

In the limit $t\to-\infty$, both expansions, (13) and (14), thus have smooth coefficients for all s. The following arguments (which are analogous to the arguments used in a similar situation in Ref. 10) show that p_j and g_j are the same for all j. Substitution of (13) into (1) gives rise to a sequence of linear first-order ordinary differential equations for the coefficients g_j. The smoothness condition at $s=0$ determines these functions unambiguously, since the corresponding homogeneous equations have solutions $f(s)^{3-7j}/f'(s)$ with strong singularities at the origin. Consequently, (13) and (14) must be the same. It is natural to suggest that the Gurevich–Pitaevskiĭ solution does indeed satisfy (4).

6. It was suggested in Ref. 2 that the asymptotic form of $v(t, x)$ in the oscillation region is described by self-similar solutions of the Whitham equations found by taking an average[11] of a cnoidal wave. The solutions were derived in Ref. 4 by the method proposed in Ref. 3. It follows from the results of Refs. 3 and 4 that the "average" consequence of (4) determines these self-similar solutions of the Whitem equations. (This point can be verified in a particularly simple way by using the results of the calculations of Ref. 7.)

7. The simultaneous satisfaction of an ordinary differential equation in x and t was established in Refs. 12 and 13 for a special solution of the nonlinear Schrödinger equation

$$-iq_t = g_{xx} + 2\delta|q|^2 q, \tag{15}$$

which describes a "phase breaking" of rapidly oscillating solutions of equations with a slight nonlinearity in several problems.[14,15] That function is an analog of the Percy integral:

$$Q = \int_R \exp\{-i(x\lambda + t\lambda^2 - \lambda^4/4)\} d\lambda, \tag{16}$$

which satisfies the linear part of the nonlinear Schrödinger equation. The integral Q is also a solution of the linear part of these ordinary differential equations in x and t, which, along with (15), satisfies its nonlinear analog $q(t, x)$ from Refs. 12–15.

Under the assumption that Eqs. (1) and (4) are simultaneously valid for the Gurevich–Pitaevskiĭ solution, we can assume that it is an analog of another integral,

$$I = \int_R \lambda \exp\{-2i(x\lambda + 4t\lambda^3 - 3456\lambda^7/35)\} d\lambda, \tag{17}$$

which satisfies the linear part of (1) and (4), as can be seen easily. (In the dissipative case, studied in Ref. 10, a nonlinear analog of the Percy function–a special solution of the Burgers equation $R_t + RR_x = R_{xx}$–was also used near the breaking point.) Both the Percy integral and the integral in (17) belong to the family of so-called special wave-catastrophe functions, which play an important role in research on the asymptotic form of solutions of linear problems.[16] We see that their analogs–special solutions of nonlinear evolution equations–arise in nonlinear problems near the breaking point. As in the linear case, they are simultaneously solutions of an ordinary differential equation.

8. The analogies between the solution of the nonlinear Schrödinger equation from Refs. 12–15 and the Percy integral and the solution of the Korteweg–de Vries equation with integral (17) under consideration here can be pursued further. It turns out that both of these special solutions belong to the class of isomonodromic solutions.[17] Isomonodromic solutions are distinguished from other solutions of the Korteweg–de Vries and nonlinear Schrödinger equations by the circumstance that the corresponding Ψ functions satisfy, in addition to the ordinary equations of the method of the inverse scattering problem,[6] linear ordinary differential equations in the spectral parameter λ with rational (in λ) coefficients. The specific form of these ordinary differential equations is determined by the behavior of the Ψ functions near their singular points in the complex λ plane. It was pointed out in Refs. 12 and 13 that in the limit $|\lambda| \to \infty$ the squares of the Ψ functions corresponding to the solution of the nonlinear Schrödinger equation discussed there "coincide" with the exponential function in the Percy integral in (16).

The same basic "coincidence" with the exponential function in the integral in (17) also occurs for the squares of the Ψ functions corresponding to solution of (1), (4). Admittedly, the equations in λ for these functions are quite unwieldy. We would nevertheless point out that the customary ordinary differential equation $\Psi_{xx}+(\lambda^2+v/6)\Psi=0$ for the potential v satisfying (4) is compatible with the equation

$$\lambda\Psi_\lambda=3456\,\{\Psi_7+7v\Psi_5/12+35v_1\Psi_4/24+[35(v_2/16+v^2/288)]\Psi_3$$

$$+35[5v_3+vv_1]\Psi_2/128+[161v_4+35(10vv_2+7(v_1)^2+v^3/3]\Psi_1/192$$

$$+[63v_5+35(vv_3+2v_1v_2+v^2v_1/6)]\Psi/384\}/5-12t\Psi_3+(x-18tv)\Psi_1-9tv_1\Psi.$$

9. Kitaev[18] has introduced a hierarchy of ordinary differential equations whose solutions are interpreted as nonlinear analogs of special wave-catastrophe functions. If we assume that the isomonodromic nature of the Gurevich–Pitaevskiĭ solution of the Korteweg–de Vries equation and the solution of the nonlinear Schrödinger equation from Refs. 12–15 has been established, then from the Kitaev standpoint[18] they can be treated as solutions of representatives of two different hierarchies of ordinary differential equations, corresponding to the case of one of the most elementary wave catastrophes: the cusp.

An even simpler case is the fold catastrophe. Among the isomonodromic special functions corresponding to this case are solutions of the nonlinear Painlevé ordinary differential equations. The isomonodromic nature makes it possible to effectively study the asymptotic behavior of the solutions of these classical equations.[19,20]

If an isomonodromic nature holds for the Gurevich–Pitaevskiĭ solution, then it can also be used to refine the asymptotic behavior in the oscillation region.

In conclusion I would like to point out that the symmetry approach which has been taken here to special nonlinear functions (whose source is an observation from Ref. 18) was developed in a study which is a collaboration with I. T. Khabibullin and which is presently being prepared for publication.

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