Matrix versions of the Calogero model

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Matrix generalizations of the N-particle quantum-mechanical Calogero model, which classify according to the representations of the symmetric group S_N, are considered. The symmetry properties of the eigenwave functions in the matrix Calogero models are analyzed.

The Calogero model is the N-particle quantum-mechanical model on a line with the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{N}[-d_i^2+x_i^2]+g\sum_{j<i}^{N}(x_i-x_j)^{-2},$$

where $d_i=\partial/\partial x_i$. It is a prime example of a solvable N-body quantum mechanical model,[1,2] which has interesting physical applications. In particular, this model is closely related to the matrix models,[3,4] while the generalized differential operators, which underlie the integrability of the model, appear in a number of different problems such as, e.g., the decoupling equations in certain formulations of conformal models[5–8] and the problem of quantization on a sphere and hyperboloid.[9] It was shown in Refs. 10–13 that the Calogero model can be interpreted as a one-dimensional reduction of the full anyon problem. It was recently suggested that[14] the Calogero model (an its trigonometric generalization) can be identified with the 2D Yang–Mills theory on a cylinder. Another intriguing link is that the higher-spin gauge theories in three[15] and four[16] space-time dimensions exhibit infinite-dimensional symmetries and higher-spin symmetries, which are described by the algebra of observables of the Calogero model.

One can speculate that the reason for all these links of the Calogero model is that the algebraic structures underlying it are as fundamental as those of the ordinary harmonic oscillator. The matrix generalizations of the Calogero model we are examining in this letter may also have useful applications.

The spectrum of the Hamiltonian H was found by Calogero.[1] The singularities of H at the planes $x_i=x_j$ force one to consider the wave functions of H separately in $N!$ distinct domains which are singled out by the sets of inequalities $x_{i_1}<x_{i_2}<...<x_{i_N}$. Performing the similarity transformation $\Psi=\beta^\nu\Phi$, where $\beta=\Pi_{x_i>x_j}(x_i-x_j)$, so that for $g=\nu(\nu-1)$ the transformed Hamiltonian $H_{\text{Cal}}=\beta^\nu H\beta^{-\nu}$ takes the form

$$H_{\text{Cal}}=-\frac{1}{2}\sum_{i=1}^{N}\left[d_i^2-x_i^2+\nu\sum_{j\neq i}\frac{2}{x_i-x_j}d_i\right], \tag{1}$$

Calogero argued[1] that for $\nu>0$ the regular eigenfunctions $H_{\text{Cal}}\Phi_n=E_n\Phi_n$ are of the form

$$\Phi_n=\phi_{nk}(r)P_k(x), \quad r^2=\frac{1}{N}\sum_{i<j}(x_i-x_j)^2,$$

where $P_k(x)$ are homogeneous polynomials of degree k, which obey the "generalized harmonic equation"

$$\left(\sum_{i=1}^{N} d_i^2+\nu\sum_{i\neq j}\frac{i}{x_i-x_j}(d_i-d_j)\right)P_k=0. \tag{2}$$

Here $\phi_{nk}(r)$ obeys a certain equation[1] which fixes the energy spectrum E_n. Calogero proved that every polynomial which obeys (2) is a symmetrical polynomial of x_i. A possible interpretation of this elegant result is that the model automatically selects the subspace of totally symmetric wave functions which extend to the whole coordinate space.

More recently, it was shown[17] how one can construct the set of eigenwave functions for the Calogero model [thus describing the solutions of (2)] with the help of the approach based on the permutation operators K_{ij} which interchange the coordinates x_i and x_j. The basic point is that by introducing

$$D_i=d_i+\nu\sum_{j\neq i}\frac{1}{x_i-x_j}(1-K_{ij}), \tag{3}$$

where K_{ij} obeys the properties $K_{ij}x_j=x_iK_{ij}$, $K_{ij}K_{jl}=K_{il}K_{ij}=K_{jl}K_{il}$, $K_{ij}=K_{ji}$, and $(K_{ij})^2=1$, we observe, first,[17,18,19] that $[D_i,D_j]=0$ and, second, that one can define[17] such creation and annihilation operators $a_i^{\mp}=1/\sqrt{2}(x_i\pm D_i)$, which obey the commutation relations

$$[a_i^{\pm},a_j^{\pm}]=0, \quad [a_i^-,a_j^+]=\delta_{ij}\left(1+\nu\sum_l K_{il}\right)-\nu K_{ij}=A_{ij}, \tag{4}$$

that the Hamiltonian $H_{\rm Univ}=\frac{1}{2}\Sigma_i\{a_i^+,a_i^-\}$ satisfies the basic property

$$[H_{\rm Univ},a_i^{\pm}]=\pm a_i^{\pm} \tag{5}$$

and turns out to be related to the original Calogero Hamiltonian (1) in a simple way:

$$H_{\rm Univ}=H_{\rm Cal}+\frac{1}{2}\nu\sum_{j\neq i}\frac{1}{(x_i-x_j)^2}(1-K_{ij}). \tag{6}$$

Based on (5) one can easily construct[17] the set of eigenwave functions of the universal Calogero Hamiltonian $H_{\rm Univ}$ (6) via the standard procedure by defining the ground state in terms of $a_i^-|0\rangle=0$, $K_{ij}|0\rangle=|0\rangle$.[1)] Since the second term on the right-hand side of (6) trivializes for totally symmetric states, we find that for this case $H_{\rm Univ}$ amounts to $H_{\rm Cal}$, thus recovering the totally symmetric wave functions of the Calogero model. It is worth noting that the universal Calogero Hamiltonian $H_{\rm Univ}$ is well defined for the wave functions with arbitrary symmetry properties, while the original Hamiltonian (1) makes sense only when it coincides with $H_{\rm Univ}$, i.e., when the second term on the right side of (6) trivializes.

The question we address in this paper is whether there exist other quantum-mechanical models which amount to the universal Calogero model (6) for subspaces of wave functions corresponding to some Young's diagram. We show that such quantum-mechanical models are described by the Hamiltonians

$$\hat{H}_{\text{Cal}}=H_{\text{Cal}}I+\frac{1}{2}\nu\sum_{i\neq j}\frac{1}{(x_i-x_j)^2}(I-T_{ij}). \tag{7}$$

Here I is the $m\times m$ unit matrix and $T_{ij}=T(p_{ij})$, where T is a $m\times m$ unitary matrix representation of the symmetric group with elementary $i\leftrightarrow j$ permutations p_{ij}. Since $T_{ij}^2=I$, each unitary matrix T_{ij} is Hermitian, which ensures hermiticity of the Hamiltonian (7).[2)] Note that both supersymmetric[13,20] and matrix[2] generalizations of the Calogero Hamiltonian considered previously correspond to the particular cases of (7) for some (reducible) representations T of the symmetric group.

The Calogero result on the symmetry property of the wave functions extends to the Hamiltonian (7) as follows: nonsingular eigenfunctions of the Hamiltonian (7) with $\nu>0$ exist if they satisfy the conditions

$$K_{ij}\Phi=T_{ij}\Phi \tag{8}$$

for all i and j, i.e., if the Hamiltonians (6) and (7) coincide.

This fact is the main result of this paper, which implies, in particular, that the condition (8), which was shown in Refs. 13 and 21 to be convenient to impose in order to solve the problem, is a kind of necessary condition which cannot be avoided. Effectively this means that the action of the symmetric group S_N on the coordinates x_i generated by K_{ij} realizes the same representation of S_N as T_{ij} do. (In particular, if $T_{ij}=1$, the wave functions turn out to be symmetrical.) The proof of this statement is as follows:

(i) it can easily be checked that if Φ (which is an m-column) is a solution of

$$\hat{H}_{\text{Cal}}\Phi=E\Phi, \tag{9}$$

then $K_{ij}T_{ij}\Phi$, and therefore $Q_{ij}=(1-K_{ij}T_{ij})\Phi$, are also some of its solutions;

(ii) multiplying each side of (9) by $(x_i-x_j)^2$ and setting $x_i=x_j$, we see that $(1-T_{ij})\Phi|_{x_i=x_j}=0$. Since K_{ij} trivializes for $x_i=x_j$, we conclude that $Q_{ij}|_{x_i=x_j}=0$.

(iii) using this conclusion, it can then be proved by analogy with the original Calogero proof that $Q_{ij}=0$. The main steps of the proof are as follows. Assume that $Q_{ij}\neq 0$. Then $Q_{ij}=(x_i-x_j)^l R_{ij}$ with some positive integer l and the column $R_{ij}|_{x_i=x_j}\neq 0$. Substituting this result into (9), taking into account (i), and analyzing the lowest-order terms in (x_i-x_j), we find

$$[l(l-1)=2\nu l-\nu(1-T_{ij})]R_{ij}|_{x_i=x_j}=0. \tag{10}$$

We now observe that $T_{ij}K_{ij}Q_{ij}-Q_{ij}$. Therefore, $T_{ij}K_{ij}R_{ij}=-(-1)^l R_{ij}$ and $T_{ij}R_{ij}|_{x_i=x_j}=-(-1)^l R_{ij}|_{x_i=x_j}$. Hence, either $\{l(l-1)+2\nu l-\nu[1+(-1)^l]\}=0$, which is impossible for $l>0$ and $\nu>0$, or $R_{ij}|_{x_i=x_j}=0$. This completes the proof.

Thus, the entire eigenwave functions of the Hamiltonian (7) can exist only when $Q_{ij}=0$, and therefore only when (8) holds. From the results of Ref. 17 it follows that such solutions of (9) do exist. Actually, let T be an irreducible representation of the symmetric group S_N, which is described by an appropriate Young's tableaux. Let $R(a_i^+)$ be an m-column of homogeneous polynomials of a_i^+ of degree k, which satisfy the condition $K_{ij}RK_{ij}^{-1}=T_{ij}R$ for all i and j. It is evident that for every m-dimensional representation T of S_N there exist such sufficiently large k that such a column $R(a_i^+)$ exists. Applied to the symmetric ground state of the Hamiltonian (6), it gives a solution of (9): $\hat{H}_{\mathrm{Cal}}R|0\rangle = H_{\mathrm{Univ}}R|0\rangle = (RH_{\mathrm{Univ}}+kR)|0\rangle=(E_0+k)R|0\rangle$.

The general structure of creation and annihilation operators for the Hamiltonian (7) is as follows. Any annihilation operator A_n is $m\times m$-matrix operator which maps the eigenfunctions of (9) with some eigenvalue E to eigenfunctions with the eigenvalue $E-n$. The matrices A_n must preserve (8); i.e., if a vector-function Φ belongs to the space of linear combinations of the eigenfunctions of (9), then $K_{ij}A_n\Phi=T_{ij}A_n\Phi$, and hence the restriction of A_n (which we will identify with A_n) to this space, satisfy the conditions

$$K_{ij}A_nK_{ij}^{-1}=T_{ij}A_nT_{ij}^{-1} \tag{11}$$

for any i and j. The defining relation $[\hat{H}_{\mathrm{Cal}},A_n]=-nA_n$, along with (11), leads to

$$[H_{\mathrm{Univ}},A_n]=-nA_n. \tag{12}$$

Since H_{Univ} is proportional to the unit matrix I, the relation (12) holds for all elments of the matrix A_n separately. One thus concludes that n-degree annihilation operators A_n for the Hamiltonian (9) are matrices which obey (11) with elements that depend on a_i and a_i^+, each having grading $-n$. We hope to present a constructive description of A_n elsewhere.

One can speculate that the models under investigation describe interactions of several groups of particles with abnormal mutual statistics. In general, various types of interacting matrix Calogero models are classified according to the irreducible representations of the symmetric group S_N.

As a simple example, let us consider the case corresponding to the Young's diagram with two rows containing $N-1$ boxes and 1 box, respectively. It is convenient to describe the space of this representation in terms of column-vectors with the components

$$\Phi_i=(1-K_{iN})F, \quad i=1,\ldots,N-1, \tag{13}$$

where F is a function which is symmetric under transpositions of the first $N-1$ variables: $K_{ij}F=F$ for $i,j=1,2,\ldots,N-1$.

The action of the operators K_{ij} on these columns is the same as the action of some $(N-1)\times(N-1)$ matrices $\tilde{T}_{ij}$, $(\tilde{K}_{ij}\Phi)_l=\Sigma_{k=1}^{N-1}(\tilde{T}_{ij})_{lk}\Phi_k$:

$$K_{ij}\Phi_l=\Phi_l, \quad \text{when} \quad i,j,l=1,2,\ldots,N-1, \quad i\neq l, \quad j\neq l$$

$$K_{ij}\Phi_j=\Phi_i, \quad \text{when} \quad i,j=1,2,\ldots,N-1$$

$$K_{iN}\Phi_i=-\Phi_i, \quad i=1,2,\ldots,N-1$$

$$K_{iN}\Phi_j=\Phi_j-\Phi_i, \quad \text{when} \quad i,j,=1,2,\ldots,N-1, \quad i\neq j. \tag{14}$$

The matrices $\widetilde{T}_{ij}$ are equivalent to unitary matrices $T_{ij}=Q^{-1}\widetilde{T}_{ij}Q$, where Q is any matrix which satisfies the conditions

$$\sum_i (Q_{ki})^2=2, \quad \sum_i (Q_{ki}-Q_{li})^2=2, \quad k\neq l. \tag{15}$$

In other words, its $N-1$ rows together with zero can be interpreted as the coordinates of the apices of a rectilinear N-hedron in $(N-1)$-dimensional space. For example, we can set $Q_{ij} = \delta_{ij} - 1/(N-1)(1+\sqrt{N})$ and $(Q^{-1})_{ij} = \delta_{ij} - 1/(N-1) \times (1+1/\sqrt{N})$.

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[1)]It should be noted that the restriction $\nu > -1/N$ leads to the vacuum $|0\rangle$ with a finite norm, and therefore the restriction $\nu>0$ guarantees the finiteness of the norm of vacuum for each N. A detailed consideration of the more subtle cases with nonnormalizable vacuums and small ν, when some singular wave functions may also exist, will be given elsewhere.

[2)]To avoid misunderstandings, we emphasize that, in contrast with K_{ij}, $m\times m$ x-independent matrices T_{ij} commute with x_i and d_i.

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