Singular Malikov–Feĭgin–Fuks vectors in topological theories

A. M. Semikhatov
P. N. Lebedev Physics Institute, Russian Academy of Sciences, 117924 Moscow, Russia

(Submitted 12 November 1993)
Pis'ma Zh. Eksp. Teor. Fiz. **58**, No. 11, 916–924 (10 December 1993)

Singular vectors with respect to the algebra of (twisted) $N=2$ supersymmetry and the Kac–Moody $sl(2)$ algebra are shown to coincide. The very simple Kazama–Suzuki model is used to derive a representation for $sl(2)$ currents in terms of a gravitation interacting with matter. A general expression for singular vectors in topological theories is derived from the Malikov–Feĭgin–Fuks formulas for $sl(2)$ singular vectors.

Associated with singular vectors in representations of infinite-dimensional algebras of symmetries of conformal theories are some irreducible models whose correlation functions obey decoupling equations, i.e., conditions expressing a decoupling of a corresponding singular vector. Much has been established about singular vectors in a representation of a Virasoro algebra ("minimal" models)[1–8] and $sl(2)$ current algebras,[9,10] and also regarding the reduction of the former to the latter.[6,8]

In this letter we examine singular vectors with respect to the algebra of $N=2$ supersymmetry in its "topological" ("twisted") version.[11,12] Topological algebras are interesting, in particular, because any such algebra with a central topological charge $c\neq 3$ allows a construction[13–17] in terms of ordinary (nonsupersymmetric) matter with a central charge $d=(c+1)(c+6)/(c-3)$, an auxiliary scalar field, and a pair of bc ghosts. Two constructions of this sort are possible,[13,14] one of which reproduces the David–Distler–Kawai (DDK) recipe[18] in the (matter) + (scalar ϕ) sector. As a result, we will call the field ϕ a "Liouville" field. There is also a second way to construct a topological algebra from matter with a central charge d, a Liouville field, and ghosts *which have spin 1* (not 2, as in the version containing DDK). We will call this version of the construction a "mirror"[14] gravitation which is interacting with matter. This construction is of interest for applications to topological singular vectors.

We denote by $|\Psi\rangle \equiv |h\rangle$ a chiral primary state[19] of a topological algebra with a $U(1)$ topological charge h. Constructing a singular vector $|\Upsilon\rangle$ on $|h\rangle$ at the l level, we substitute into it the expressions for the topological generators in terms of the "components" of the matter, the Liouville field, and the ghosts. We now require a decoupling of the singular vector, i.e., the vanishing of all correlation functions of the type $\langle \Upsilon(z_a)\Pi_{b\neq a}\Psi_b(z_b)\rangle$, where Ψ_b are chiral primary fields. The field Ψ is a chiral primary field if we have $|\Psi\rangle = |\text{matter}\rangle \otimes |\text{Liouville}\rangle \otimes |0\rangle_{bc}$, where $|0\rangle_{bc}$ is the ghost vacuum, and $|\text{matter}\rangle \otimes |\text{Liouville}\rangle$ is the primary state of the matter, which is determined by a dressed Liouville scalar. A reduction to the matter + Liouville theory has thus been defined well. The image of the topological singular vectors in the course of this reduction is the same as the result of a direct construction of singular vectors in the matter + Liouville theory which satisfy the Kontsevich–Miva dressing conditions.[14,20] The conditions for their decoupling are written as partial differential equations of order l for the correlation functions. It turns out[14,20] that the differential operators which specify these equations factorize (completely or within a certain obstacle) in terms of Virasoro generators $L_{\geq -1}$ written in terms of the times t_r introduced by the Miva transformation $t_r = (1/r)\,\Sigma_b n_b z_b^{-r}$, where $r \geq 1$, and n_b are Liouville charges of the fields Ψ_b. A complete factorization occurs for singular vectors of the $(l,1)$ or $(1,l)$ type, described below.

The two integers $l' \equiv 2j_1+1$ and $l'' \equiv 2j_2+1$ characterize a "topological" singular vector $|\Upsilon\rangle$ in the following way: $|\Upsilon\rangle$ is constructed at the $l = l'l''$ level on the prime chiral field $|\mathrm{h}\rangle$, the $U(1)$ topological charge h of which is related to the topological central charge with the equality

$$\mathrm{h} = \frac{\mathrm{c}-3}{3}\, j_1 + 2\, j_2 \tag{1}$$

with integer or half-integer j_1 and j_2. For vectors of the (l', l'') type, in which neither number is equal to 1, there is an obstacle to complete factorization.[14]

This case is reminiscent of the "good" behavior of singular vectors of the $(2j_1+1,1)$ type in the Kac–Moody $sl(2)$ algebra (cf. Refs. 5, 6, and 8). A direct comparison with that algebra turns out to be possible if we use the (twisted) Kazama–Suzuki model[21] $\widetilde{sl}(2)_k \oplus u(1)/u(1)$, where $sl(2)_k$ denotes the $sl(2)$ algebra of level k with the energy–momentum tensor

$$\widetilde{T}^S = \frac{1}{k+2}\left(J^0 J^0 - \frac{1}{2}\,(J^+J^- + J^-J^+)\right) + \partial J^0. \tag{2}$$

The $u(1)$ algebra in the numerator of the Kazama–Suzuki model fermionizes into a pair of spin-1 ghosts, which we will designate below as BC (cf. Ref. 16). The topological algebra is then constructed by a known method.[11,22–24] The odd generators $\mathcal{Q}$ and $\mathcal{G}$ are given by

$$\mathcal{Q} = \sqrt{\frac{2}{k+2}}\, BJ^+, \quad \mathcal{G} = -\sqrt{\frac{2}{k+2}}\, CJ^-, \tag{3}$$

and the $U(1)$ topological current and the energy–momentum tensor are realized in the form

$$\mathcal{H} = -\frac{k}{k+2}\, BC - \frac{2}{k+2}\, J^0, \quad \mathcal{T} = -\frac{1}{(k+2)}(J^+J^-) + \frac{k}{k+2}\, \partial B \cdot C + \frac{2}{k+2}\, BCJ^0. \tag{4}$$

The central topological charge turns out to be $\mathrm{c} = (3k/k+2)$. We thus have the mapping $\mathrm{t} \to \mathcal{U}[sl(2)_k \oplus u(1)]$, where t is the topological algebra, and $\mathcal{U}$ means a universal envelope. The representation of the topological algebra t as matter m dressed by a "mirror" gravitation allows us to write $\mathrm{t} = \mathrm{m} \oplus \mathrm{l} \oplus [bc]$, where l and $[bc]$ are the Liouville and ghost theories, respectively. We then have the short, exact sequences (we are omitting $\mathcal{U}$)

$$
\begin{array}{c}
0 \\
\downarrow \\
\mathrm{m} \oplus \mathrm{I} \oplus [bc] \\
\downarrow \\
0 \to sl(2) \to \mathscr{A} \to u(1)_{BC} \to 0, \\
\downarrow \\
u(1)_v \\
\downarrow \\
0
\end{array}
\tag{5}
$$

where $u(1)_v$ is the "denominator" in the Kazama–Suzuki model, i.e., the algebra which is generated by the current

$$\partial v = \sqrt{\frac{2}{k+2}}\,(J^0 - BC) \tag{6}$$

[and which is decoupled from generators (3) and (4)]. Not only the horizontal exact sequence but also the vertical one split up, giving us two methods for describing the algebra $\mathscr{A}$. First, we have simply $sl(2) \oplus u(1)$, but at the same time this is $\mathrm{m} \oplus \mathrm{I} \oplus [bc] \oplus u(1)_v$, and in these terms the splitting of the horizontal sequence occurs through a representation of "Kazama–Suzuki" ghosts in the form

$$B = be^{-\sqrt{\frac{2}{k+2}}(v-\phi)}, \quad C = ce^{\sqrt{\frac{2}{k+2}}(v-\phi)}. \tag{7}$$

The immersion $sl(2) \hookrightarrow \mathrm{m} \oplus \mathrm{I} \oplus [bc] \oplus u(1)_v$, on the other hand, is specified by the following explicit formulas:

$$
\begin{aligned}
&J^+ = e^{\sqrt{\frac{2}{k+2}}(v-\phi)}, \quad J^0 = -i + \sqrt{\frac{2}{k+2}}\,I + \frac{k}{\sqrt{2(k+2)}}\,\partial v, \\
&J^- = \{-(k+2)(T+T_{\mathrm{L}}) + i^2 - (k+1)\partial i - \sqrt{2(k+2)}\,I \cdot i\}e^{\sqrt{\frac{2}{k+2}}(\phi - v)},
\end{aligned}
\tag{8}
$$

where $i = -bc$ is a ghost current, $\partial\phi = I$ is a Liouville current, and T and T_{L} are the energy–momentum tensors of the matter and the Liouville field.

Construction (8) can be thought of as a version of a Wakimoto bosonization.[25] The obvious redundance of the number of fields can be eliminated by a transformation which explicitly decouples the BC current:

$$\partial F = i + \sqrt{\frac{2}{k+2}}\,(\partial v - \partial\phi). \tag{9}$$

Along with ∂F we introduce several new independent fields, by means of

$$\partial\chi=\sqrt{\frac{k+2}{k}}\,\partial\phi-\sqrt{\frac{2}{k}}\,i,\quad \partial\psi=\sqrt{\frac{k}{k+2}}\,\partial v+\frac{2}{\sqrt{k(k+2)}}\,\partial\phi-\sqrt{\frac{2}{k}}\,i \tag{10}$$

(with the respective signatures + and −). The $sl(2)$ currents in (8) now take the form

$$J^{+}=e^{\sqrt{\frac{2}{k}}(\psi-\chi)},\quad J^{0}=\sqrt{\frac{k}{2}}\,\partial\psi,$$
$$J^{-}=\left[-(k+2)T+k\left(\frac{1}{2}\,\partial\chi\partial\chi+\frac{k+1}{\sqrt{2k}}\,\partial^{2}\chi\right)\right]e^{-\sqrt{2/k}(\psi-\chi)}. \tag{11}$$

Both (8) and (11) thus contain arbitrary matter with a central charge d, for which *it is not required* that bosonization be assumed, since the matter enters exclusively through its own Virasoro generators.

We will now use the mappings constructed above for an *explicit calculation* of the singular vectors. Specifically, we will show that the *singular vectors of the Kac–Moody $sl(2)$ algebra coincide with topological singular vectors.* We recall that the singular vectors $sl(2)_k$, which are specified by two integers r and s, can be written in Malikov–Feĭgin–Fuks form:[10]

$$|\mathrm{MFF}rs\rangle=(J_0^-)^{r+(s-1)(k+2)}(J_{-1}^+)^{r+(s-2)(k+2)}(J_0^-)^{r+(s-3)(k+2)}\ldots$$
$$\ldots(J_0^-)^{r-(s-1)(k+2)}\left|\left\{\frac{r-1}{2},\frac{s-1}{2}\right\}\right\rangle, \tag{12}$$

where $|\{j_1,j_2\}\rangle$ is the $sl(2)$ state of the senior weight with an eigenvalue J^0 equal to

$$j=j_1-j_2(k+2). \tag{13}$$

Expression (12) is not as innocent as it may appear: It is simple only in the $s=1$ case [reducing in this case to $(J_0^-)^r$], while the $|\mathrm{MFF}14\rangle$ state, for example, is written as a *polynomial* in currents consisting of 19 different terms, and $|\mathrm{MFF}15\rangle$ consists of 42 terms. To make the $|\mathrm{MFF}rs\rangle$ states J^0-neutral, and to also make them states of the $l=rs$ level, we need to multiply them from the right by $(J_{-1}^+)^r$.

We will now demonstrate the appearance of exactly the same vectors in topological theories. To save space here, we restrict the examples to level 4, but these examples are fairly representative, since (in contrast with, say, the 2, 3, and 5 levels) at the $4=2\cdot2$ level we are not restricted to "good" ("thermal" in the terminology of Ref. 8) singular vectors. Specifically, the $\{j_1,j_2\}$ pair takes on the values {3/2, 0}, {1/2, 1/2}, and {0, 3/2}, so we have three possibilities:

$$\left\{\frac{3}{2},\,0\right\},\quad \mathrm{h}=\frac{1}{2}(\mathrm{c}-3),\qquad \left\{\frac{1}{2},\,\frac{1}{2}\right\},\quad \mathrm{h}=\frac{1}{6}(\mathrm{c}+3),\qquad \left\{0,\,\frac{3}{2}\right\},\quad \mathrm{h}=3. \tag{14}$$

Correspondingly, we have the following singular states.

In the first case, with $\{j_1,j_2\}\{3/2,0\}$ and $\mathrm{h}=(\mathrm{c}-3)/2$, we have the singular vector

$$
\begin{aligned}
&\left|\Upsilon\left[\begin{smallmatrix}3\\2\end{smallmatrix}\cdot 0\right]\right\rangle\\
&=\frac{108}{\tilde{c}^2}\Bigg(\frac{72(5-2c)}{\tilde{c}^2}\mathscr{H}_{-4}+\frac{2(18+12c-11c^2+c^3)}{\tilde{c}^2}\mathscr{L}_{-4}+\frac{3(2c-13)}{\tilde{c}}\mathscr{G}_{-3}\mathscr{Q}_{-1}\\
&\quad-\frac{2(51-13c+2c^2)}{\tilde{c}^2}\mathscr{H}_{-3}\mathscr{L}_{-1}+\frac{6(1-c)}{\tilde{c}}\mathscr{H}_{-2}\mathscr{L}_{-2}\\
&\quad+\frac{6(13-2c)}{\tilde{c}}\mathscr{L}_{-3}\mathscr{H}_{-1}+\frac{4(21-16c+2c^2)}{\tilde{c}^2}\mathscr{L}_{-3}\mathscr{L}_{-1}\\
&\quad+\frac{225+13c-2c^2}{\tilde{c}^2}\mathscr{Q}_{-3}\mathscr{G}_{-1}-\frac{3(33-16c+c^2)}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{G}_{-2}\\
&\quad-\frac{36}{\tilde{c}}\mathscr{G}_{-2}\mathscr{H}_{-1}\mathscr{Q}_{-1}+\frac{18(c-2)}{\tilde{c}^2}\mathscr{G}_{-2}\mathscr{L}_{-1}\mathscr{Q}_{-1}+3\mathscr{L}^2_{-2}\\
&\quad+\frac{12(3c-2)}{\tilde{c}^2}\mathscr{H}_{-2}\mathscr{H}_{-1}\mathscr{L}_{-1}+\frac{4(3-5c)}{\tilde{c}^2}\mathscr{H}_{-2}\mathscr{L}^2_{-1}+\frac{12}{\tilde{c}^2}\mathscr{L}^4_{-1}\\
&\quad+\frac{6(3c-2)}{\tilde{c}^2}\mathscr{H}_{-2}\mathscr{Q}_{-1}\mathscr{G}_{-1}+\frac{36}{\tilde{c}}\mathscr{L}_{-2}\mathscr{H}^2_{-1}-\frac{60}{\tilde{c}}\mathscr{L}_{-2}\mathscr{H}_{-1}\mathscr{L}_{-1}\\
&\quad-\frac{36}{\tilde{c}^2}\mathscr{L}^2_{-1}\mathscr{Q}_{-1}\mathscr{G}_{-1}+\frac{20}{\tilde{c}}\mathscr{L}_{-2}\mathscr{L}^2_{-1}+\frac{6(9-2c)}{\tilde{c}^2}\mathscr{L}_{-2}\mathscr{Q}_{-1}\mathscr{G}_{-1}\\
&\quad-\frac{18(14-c)}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{H}_{-1}\mathscr{G}_{-1}+\frac{2(72-5c)}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{L}_{-1}\mathscr{G}_{-1}\\
&\quad-\frac{72}{\tilde{c}^2}\mathscr{H}^3_{-1}\mathscr{L}_{-1}+\frac{132}{\tilde{c}^2}\mathscr{H}^2_{-1}\mathscr{L}^2_{-1}\\
&\quad-\frac{108}{\tilde{c}^2}\mathscr{H}^2_{-1}\mathscr{Q}_{-1}\mathscr{G}_{-1}-\frac{72}{\tilde{c}^2}\mathscr{H}_{-1}\mathscr{L}^3_{-1}\\
&\quad+\frac{132}{\tilde{c}^2}\mathscr{H}_{-1}\mathscr{L}_{-1}\mathscr{Q}_{-1}\mathscr{G}_{-1}\Bigg)\,|(c-3)/2\rangle, \qquad (15)
\end{aligned}
$$

where for brevity we have used the notation $\tilde{c}=c-3$. Furthermore, for $\{j_1, j_2\}=\{1/2, 1/2\}$, $h=(c+3)/6$, the topological singular vector is

$$|\Upsilon[\tfrac{1}{2},\tfrac{1}{2}]\rangle$$

$$=\frac{36}{\tilde{c}^4}\Bigg(\frac{72(-45+6c-5c^2)}{\tilde{c}^2}\mathscr{H}_{-4}-\frac{12(135-9c+15c^2-c^3)}{\tilde{c}^2}\mathscr{L}_{-4}$$

$$-\frac{3(135+45c-3c^2-c^3)}{\tilde{c}^2}\mathscr{G}_{-3}\mathscr{Q}_{-1}+\frac{6(9-30c+c^2)}{\tilde{c}}\mathscr{H}_{-3}\mathscr{L}_{-1}$$

$$+\frac{6(-135-27c+3c^2-c^3)}{\tilde{c}^2}\mathscr{H}_{-2}\mathscr{L}_{-2}+\frac{6(135+45c-3c^2-c^3)}{\tilde{c}^2}\mathscr{L}_{-3}\mathscr{H}_{-1}$$

$$+\frac{12(-81-9c-3c^2+c^3)}{\tilde{c}^2}\mathscr{L}_{-3}\mathscr{L}_{-1}+\frac{(-27-6c+c^2)^2}{\tilde{c}^2}\mathscr{L}^2_{-2}$$

$$+\frac{3(-135+315c+3c^2+c^3)}{\tilde{c}^2}\mathscr{Q}_{-3}\mathscr{G}_{-1}+\frac{3(135-63c+33c^2-c^3)}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{G}_{-2}$$

$$-\frac{36(3+c)^2}{\tilde{c}^2}\mathscr{G}_{-2}\mathscr{H}_{-1}\mathscr{Q}_{-1}+\frac{72(9-3c+c^2)}{\tilde{c}^2}\mathscr{G}_{-2}\mathscr{L}_{-1}\mathscr{Q}_{-1}$$

$$+\frac{72(18+3c+c^2)}{\tilde{c}^2}\mathscr{H}_{-2}\mathscr{H}_{-1}\mathscr{L}_{-1}-\frac{108(3+c)}{\tilde{c}}\mathscr{H}_{-2}\mathscr{L}^2_{-1}$$

$$+\frac{36(18+3c+c^2)}{\tilde{c}^2}\mathscr{H}_{-2}\mathscr{Q}_{-1}\mathscr{G}_{-1}+\frac{36(3+c)^2}{\tilde{c}^2}\mathscr{L}_{-2}\mathscr{H}^2_{-1}$$

$$+\frac{12(-135-27c+3c^2-c^3)}{\tilde{c}^2}\mathscr{L}_{-2}\mathscr{H}_{-1}\mathscr{L}_{-1}+\frac{12(45-6c+c^2)}{\tilde{c}}\mathscr{L}_{-2}\mathscr{L}^2_{-1}$$

$$+\frac{6(-27-63c+15c^2-c^3)}{\tilde{c}^2}\mathscr{L}_{-2}\mathscr{Q}_{-1}\mathscr{G}_{-1}-\frac{648c}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{H}_{-1}\mathscr{G}_{-1}$$

$$+\frac{36(-9+6c+c^2)}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{L}_{-1}\mathscr{G}_{-1}-\frac{216(3+c)}{\tilde{c}^2}\mathscr{H}^3_{-1}\mathscr{L}_{-1}$$

$$-\frac{36(9-12c-c^2)}{\tilde{c}^2}\mathscr{H}^2_{-1}\mathscr{L}^2_{-1}-\frac{324(3+c)}{\tilde{c}^2}(\mathscr{H}_{-1})^2\mathscr{Q}_{-1}\mathscr{G}_{-1}$$

$$+36\mathscr{L}^4_{-1}-\frac{72(3+c)}{\tilde{c}}\mathscr{H}_{-1}\mathscr{L}^3_{-1}-\frac{36(9-12c-c^2)}{\tilde{c}^2}\mathscr{H}_{-1}\mathscr{L}_{-1}\mathscr{Q}_{-1}\mathscr{G}_{-1}$$

$$-\frac{36(3+c)}{\tilde{c}}\mathscr{L}^2_{-1}\mathscr{Q}_{-1}\mathscr{G}_{-1}\Bigg)\left|\frac{c+3}{6}\right\rangle. \tag{16}$$

Finally, for $\{j_1, j_2\}=\{0, 3/2\}$, h=3 we have the singular vector

$$|\Upsilon|_{0,\frac{3}{2}}|\rangle=\frac{1296}{\tilde{c}^4}\Bigg(-\frac{216(15+7c)}{\tilde{c}^3}\mathscr{H}_{-4}+\frac{36(-15+c)}{\tilde{c}^2}\mathscr{L}_{-4}+\frac{270}{\tilde{c}^2}\mathscr{G}_{-3}\mathscr{Q}_{-1}$$

$$-\frac{12(207+c^2)}{\tilde{c}^3}\mathscr{H}_{-3}\mathscr{L}_{-1}-\frac{216(15+c)}{\tilde{c}}{}^3\mathscr{H}_{-2}\mathscr{L}_{-2}-\frac{540}{\tilde{c}^2}\mathscr{L}_{-3}\mathscr{H}_{-1}$$

$$+\frac{12(-6+5c)}{\tilde{c}^2}\mathscr{L}_{-3}\mathscr{L}_{-1}+\frac{324}{\tilde{c}^2}(\mathscr{L}_{-2})^2+\frac{30(9+c)}{\tilde{c}^2}\mathscr{Q}_{-3}\mathscr{G}_{-1}$$

$$+\frac{270}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{G}_{-2}-\frac{3888}{\tilde{c}^3}\mathscr{G}_{-2}\mathscr{H}_{-1}\mathscr{Q}_{-1}+\frac{378}{\tilde{c}^2}\mathscr{G}_{-2}\mathscr{L}_{-1}\mathscr{Q}_{-1}$$

$$+\frac{36(87+7c)}{\tilde{c}^3}\mathscr{H}_{-2}\mathscr{H}_{-1}\mathscr{L}_{-1}+\mathscr{L}^4_{-1}-\frac{24(12+c)}{\tilde{c}^2}\mathscr{H}_{-2}\mathscr{L}^2_{-1}$$

$$+\frac{18(87+7c)}{\tilde{c}^3}\mathscr{H}_{-2}\mathscr{Q}_{-1}\mathscr{G}_{-1}+\frac{3888}{\tilde{c}}{}^3\mathscr{L}_{-2}\mathscr{H}^2_{-1}-\frac{1080}{\tilde{c}^2}\mathscr{L}_{-2}\mathscr{H}_{-1}\mathscr{L}_{-1}$$

$$+\frac{60}{\tilde{c}}\mathscr{L}_{-2}\mathscr{L}^2_{-1}-\frac{162}{\tilde{c}^2}\mathscr{L}_{-2}\mathscr{Q}_{-1}\mathscr{G}_{-1}-\frac{270}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{H}_{-1}\mathscr{G}_{-1}$$

$$-\frac{6(9-4c)}{\tilde{c}^2}\mathscr{Q}_{-2}\mathscr{L}_{-1}\mathscr{G}_{-1}-\frac{1296}{\tilde{c}^3}\mathscr{H}^3_{-1}\mathscr{L}_{-1}+\frac{396}{\tilde{c}^2}\mathscr{H}^2_{-1}\mathscr{L}^2_{-1}$$

$$-\frac{1944}{\tilde{c}^3}\mathscr{H}^2_{-1}\mathscr{Q}_{-1}\mathscr{G}_{-1}-\frac{36}{\tilde{c}}\mathscr{H}_{-1}\mathscr{L}^3_{-1}+\frac{396}{\tilde{c}^2}\mathscr{H}_{-1}\mathscr{L}_{-1}\mathscr{Q}_{-1}\mathscr{G}_{-1}$$

$$-\frac{18}{\tilde{c}}\mathscr{L}^2_{-1}\mathscr{Q}_{-1}\mathscr{G}_{-1}\Bigg)|3. \qquad (17)$$

Let us consider the singular vectors in the Kazama–Suzuki parametrization. We have the following expansion for the energy–momentum tensors and for the prime states:

$$\tilde{T}^S-B\partial C=\mathscr{T}+\frac{1}{2}\partial v\partial v+\frac{k+1}{\sqrt{2(k+2)}}\partial^2 v,|\ \{j_1,j_2\}\rangle\otimes|\ 0\rangle_{BC}=|h_j\rangle\otimes|V_j\rangle, \qquad (18)$$

where $\mathscr{T}$ is the topological energy–momentum tensor, h_j is expressed in terms of $\{j_1,j_2\}$ by means of Eq. (1), and we have $V_j=\exp n_j v$, where $n_j=j\sqrt{(2/k+2)}$. The background charge of the matter, $Q=\sqrt{(1\text{-}d)/3}$, is now equal to $Q=[\sqrt{2}(k+1)/\sqrt{k+2}]$. From (4) and the conditions for the senior weight we find $h=h_j=-[2_j/(k+2)]$. We can thus rewrite (1) in the form of parametrization (13) for the senior weight of the $sl(2)$ affine algebra.

Reading Eqs. (18) from right to left, we find that the topological singular states $|\Upsilon\rangle$ are calculated in terms of the Kac–Moody algebra $sl(2)$ in the following way:

$$|\Upsilon\rangle\otimes|V\rangle=|S\rangle_{sl(2)}\otimes|0\rangle_{BC}, \qquad (19)$$

where *the right side does not have BC oscillators, and the* $|S\rangle$ *states are singular vectors of sl(2) algebra.* We will go through the general arguments in another place; in the present letter we report an explicit test of the coincidence of the $|S\rangle$ states with the vectors $|\mathrm{MFF}rs\rangle$ from (12) at the 2, 3, and 4 levels. In the process we find a useful new expression for the Malikov–Feĭgin–Fuks states in a form which uses the Sugawara energy–momentum tensor (2), which is *not* explicitly present in the original formulation, (12), but which is nevertheless necessary for writing the corresponding decoupling equations as *differential* equations. More precisely, the operators found on the left sides of these equations are differential operators with values in the $sl(2)$ Lie algebra. There are indications that they are extremely closely related to certain other expansions of differential operators which were recently used in Refs. 27 and 28.

Continuing our example of vectors at level 4, we find by direct (although somewhat tedious) calculations that state (15) generates the following $sl(2)$ state in accordance with (19):

$$\begin{aligned}|S_{\left|\frac{3}{2},0\right|}\rangle=&(k+2)[3(-112-102k-11k^2+3k^3)J^0_{-4}+3(8+20k+3k^2-k^3)\tilde{L}^S_{-4}\\&-3(6-4k+k^2)J^-_{-3}J^+_{-1}+3(4-k)(k+2)J^-_{-2}J^+_{-2}-3(96+59k\\&+4k^2)J^0_{-3}J^0_{-1}+3(k-4)(k+2)(7+2k)J^0_{-3}\tilde{L}^S_{-1}-3(6+7k-k^2)J^0_{-2}J^0_{-2}\\&+(4-k)(11+3k)J^+_{-4}J^-_0+3(22+6k-k^2)J^+_{-3}J^-_{-1}+9(k+2)\\&\times(4+k)\tilde{L}^S_{-3}J^0_{-1}+(k+2)(28-4k-3k^2)\tilde{L}^S_{-3}\tilde{L}^S_{-1}+9(k+2)(\tilde{L}^S_{-2})^2+9\\&\times(4+k)J^-_{-2}J^0_{-1}J^+_{-1}+3(4-k)(k+2)J^-_{-2}J^+_{-1}\tilde{L}^S_{-1}-18(4+k)J^0_{-2}J^0_{-1}J^0_{-1}\\&+6(-4+k)(k+2)J^0_{-2}J^0_{-1}\tilde{L}^S_{-1}+(28+26k+3k^2)J^0_{-2}J^+_{-2}J^-_0\\&+(32+13k)J^+_{-3}J^0_{-1}J^-_0+(k+2)(8+5k)J^+_{-3}\tilde{L}^S_{-1}J^-_0-3(k+2)\\&\times(6+k)J^+_{-2}\tilde{L}^S_{-2}J^-_0-10(k+2)^2\tilde{L}^S_{-2}(\tilde{L}^S_{-1})^2+3(4+k)J^+_{-2}J^0_{-1}J^0_{-1}J^-_0\\&+8(k+2)J^+_{-2}J^0_{-1}\tilde{L}^S_{-1}J^-_0+6(k+2)^2J^+_{-2}(\tilde{L}^S_{-1})^2J^-_0\\&+(k+2)^3(\tilde{L}^S_{-1})^4]\left|\left\{\frac{3}{2},0\right\}\right\rangle,\end{aligned}\tag{20}$$

where $\tilde{L}^S_m$ are modes of the twisted Sugawara energy–momentum tensor. Now substituting the explicit expressions for $\tilde{L}^S_m$ in terms of the currents, we can evaluate (20) in the form

$$|\ S_{\left|\frac{3}{2},0\right|}\rangle=J^+_{-1}J^+_{-1}J^+_{-1}J^+_{-1}J^-_0J^-_0J^-_0J^-_0\left|\left\{\frac{3}{2},0\right\}\right\rangle=J^+_{-1}J^+_{-1}J^+_{-1}J^+_{-1}|\mathrm{MFF41}\rangle.\tag{21}$$

The simplifications are equally striking in all other cases. In the first place, the modes of *BC* ghosts in expressions of type (20) cancel out in a nontrivial way when they are found from topological singular vectors (these expressions, including the Sugawara energy–momentum tensor, will be written in a separate publication). Even further, in substituting the expressions for the modes of the Sugawara energy–momentum tensor in terms of currents, we observe a coincidence of the topological singular vectors with Malikov–Feĭgin–Fuks vectors (12). For the topological singular vectors discussed above, with $\{j_1, j_2\}=\{1/2, 1/2\}$ and $\{j_1, j_2\}=\{0, (3/2)\}$, for example, the reader can verify that the $sl(2)$ states $| S\rangle$ calculated from (19) are of the form

$$|S_{\left|\frac{1}{2},\frac{1}{2}\right|}\rangle=J^+_{-1}J^+_{-1}|\ \mathrm{MFF22}\rangle,\quad |S_{\left|0,\frac{3}{2}\right|}\rangle=J^+_{-1}|\ \mathrm{MFF14}\rangle. \tag{22}$$

The normalizations adopted above for the *topological* singular vectors were chosen specifically for exact coincidence in these formulas. In this manner, however, we eliminate the value of the topological central charge $c=3$. For this value, the construction which represents a topological theory as dressed gravitational matter does not work. Nevertheless, from the standpoint of the topological algebra itself the value $c=3$ is nothing special, and singular vectors can be continued smoothly to this point. For this purpose, it is sufficient to alter the normalization of the singular vectors written above, multiplying them by the smallest necessary power of $(c-3)$ and setting $c=3$. From the standpoint of the $sl(2)$ theory, $c\to 3$ corresponds to $|k|\to\infty$, and one can see explicitly how Kazama–Suzuki mappings (3) and (4) degenerate as $|k|\to\infty$: For this purpose we need a renormalization of the currents by means of $J^{0,\pm}\mapsto\sqrt{k+2}J^{0,\pm}$. With $|k|=\infty$ we are then left with a complex scalar current $J^{\pm}$ and an independent current J^0, which is decoupled from the $c=3$ topological algebra, while the algebra itself is now constructed from $J^{\pm}$and BC ghosts (the latter are not renormalized). For $c=3$, the topological singular states can be thought of as "permission" for a Malikov–Feĭgin–Fuks construction in the limit $k\to\infty$. In the original representation, (12), on the other hand, this limit appears poorly defined; one can calculate it in the "Sugawara" form of the singular vectors as in (20).

We have thus established the coincidence of the topological and $sl(2)$ singular vectors. In this manner, the different entities in diagram (5) have common singular vectors, which can be rewritten in a variety of ways. In particular, along with the ways we have used here it is possible to carry out calculations with the help of Eqs. (8): Since for $sl(2)$ singular vectors there is an "explicit" Eq. (12) (with the stipulations following it), by substituting into it expressions (8) [or their "irreducible" form (11)] we find a general formula for topological singular vectors corresponding interpretation of the topological theory as a correspondingly dressed matter. In an interpretation of topological theory in the form $\mathfrak{t}=\mathfrak{m}\oplus\mathfrak{l}\oplus[\mathrm{bc}]$, chiral primary fields can be represented by the following ghost-independent operators:

$$\Psi=e^{j\sqrt{\frac{2}{k+2}}(v-\phi)}\psi, \tag{23}$$

where ψ is a primary state of the matter. The operator products of the type $J^-(z)\cdot\Psi(w)$ can now be evaluated directly. The merging of the two exponential functions involved here evidently does not give rise to poles, while with respect to the energy–momentum tensors appearing in J^- the field Ψ has a *zero* dimensionality.

By interpreting the topological theory as matter dressed by a ("mirror") gravitation, we can probably reach an understanding of the appearance of the $sl(2)$ algebra as a covariant version of Ref. 26 (in particular, one which includes ghosts).

I would like to take this opportunity to thank W. Lerche for numerous discussions.

[1] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).
[2] B. L. Feĭgin and L. B. Fuks, Funk. An. Pril. **16**, 47 (1982).
[3] D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984).
[4] Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B **240**, 312 (1984).
[5] L. Benoit and Y. Saint-Aubin, Phys. Lett. B **215**, 517 (1988).
[6] M. Bauer, Ph. Di. Francesco, C. Itzykson *et al.*, Nucl. Phys. B **362**, 515 (1991).
[7] M. Bauer and N. Sochen, Phys. Lett. B **275**, 82 (1992).
[8] A. Ch. Ganchev and V. B. Petkova, Phys. Lett. B **293**, 56 (1992); F. Furlan, A. Ch. Ganchev, R. Paunov *et al.*, "Solutions of the Knizhnik–Zamolodchikov equations with rational isospins, the reduction to minimal models," Preprint.
[9] V. G. Kač and D. A. Kazhdan, Adv. Math. **34**, 97 (1979).
[10] F. G. Malikov, B. L. Feĭgin, and D. B. Fuks, Funk. An. Pril. **20**, 25 (1987).
[11] T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A **4**, 1653 (1990).
[12] E. Witten, Commun. Math. Phys. **118**, 411 (1988); Nucl. Phys. B **340**, 281 (1990).
[13] B. Gato-Rivera and A. M. Semikhatov, Phys. Lett. B **288**, 38 (1992).
[14] B Gato-Rivera and A. M. Semikhatov, Nucl. Phys. B **408**, 133 (1993).
[15] M. Bershadsky, W. Lerche, D. Nemeschansky *et al.*, Nucl. Phys. B **401**, 304 (1993).
[16] S. Mukhi and C. Vafa, Preprint HUTP-93/A002.
[17] A. Giveon and M. Roček, Preprint NSF-ITP-93-17 (February 1993).
[18] F. David, Mod. Phys. Lett. A **3**, 1615 (1988); J. Distler and H. Kawai, Nucl. Phys. B **321**, 509 (1989).
[19] W. Lerche, C. Vafa, and N. P. Warner, Nucl. Phys. B **324**, 427 (1989).
[20] A. M. Semikhatov, Nucl. Phys. B **386**, 139 (1992).
[21] Y. Kazama and H. Suzuki, Nucl. Phys. B **321**, 232 (1989).
[22] T. Eguchi, S. Hosono, and S.-K. Yang, Commun. Math. Phys. **140**, 159 (1991).
[23] W. Lerche, Phys. Lett. B **252**, 349 (1990).
[24] T. Nakatsu and Y. Sugawara, Nucl. Phys. B **385**, 276 (1992).
[25] M. Wakimoto, Commun. Math. Phys. **104**, 604 (1986); Vl. S. Dotsenko, Nucl. Phys. B **338**, 747 (1990).
[26] A. M. Polyakov, Mod. Phys. Lett. A **2**, 893 (1987).
[27] L. Brink, T. H. Hansson, and M. A. Vasiliev, Phys. Lett. B **286**, 109 (1992).
[28] G. Felder and A. Veselov, "Shift operators for the quantum Calogero–Sutherland problems via Knizhnik–Zamolodchikov equation," ETH Preprint (1993).

Translated by D. Parsons