Causality and Källen–Lehmann-like representation of the fermionic string propagator[1)]

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A spectral representation for the Green's functions of the free fermionic Ramond–Neveu–Schwarz string is obtained. This representation is valid for any initial and final string configurations and it manifests an exponential growth of spectral densities intrinsic in nonlocalizable theories. The causal and singular properties of the propagators are considered.

The Källen–Lehmann-like representation for string propagators with point-like boundary conditions was obtained in Refs. 1 and 2, starting with the path integral approach.[3] The case of any initial and final string configurations for bosonic strings was considered in Ref. 4 by means of the operator formalism. As expected,[5] in all these cases the growth of the corresponding spectral densities of the propagators turns out to be linear exponential with the fundamental length $l = 2\pi\sqrt{\alpha'}$. The field theories with such a growth of matrix elements are nonlocalizable. These theories were investigated in Refs. 6 and 7.

The purpose of the present work is to derive such a spectral representation for the fermionic Ramond–Neveu–Schwarz string propagators and to show that causal and singular properties of the propagators do not depend on boundary conditions.

Let us first consider the NS sector. We will follow the notation of Ref. 8 and use a system of units for which $2\alpha' = 1$.

Let us proceed from the following BRST invariant expression for the Hamiltonian of the open fermionic string in the NS sector:

$$\hat{L}_0=\frac{1}{2}\hat{p}^2+\sum_{n=1}^{\infty} n(\hat{a}_{-n}\hat{a}_n+\hat{c}_{-n}\hat{b}_n+\hat{b}_{-n}\hat{c}_n)+\sum_{r=\frac{1}{2}}^{\infty} r(\hat{\psi}_{-r}\hat{\psi}_r+\hat{\beta}_{-r}\hat{\gamma}_r-\hat{\gamma}_{-r}\hat{\psi}_r)-\frac{1}{2}, \qquad (1)$$

where the operators obey the commutation relations

$$[\hat{x}^\mu,\hat{p}^\nu]_-=i\delta^{\mu\nu} \quad [\hat{a}^\mu_m,\hat{a}^\nu_n]_-=\mathrm{sign}(m)\delta_{m+n}\delta^{\mu\nu}, \quad [\hat{\psi}^\mu_r,\hat{\psi}^\nu_s]_+=\delta_{r+s}\delta^{\mu\nu},$$

$$[\hat{c}_m,\hat{b}_n]_+=\delta_{m+n}, \quad [\hat{\gamma}_r,\hat{\beta}_s]_-=\delta_{r+s}, \quad \mu,\nu=1,\ldots,D=10. \qquad (2)$$

Here the other (anti)commutators are equal to zero.

We use a picture in which all the operators with the negative (positive) indices are the creation (annihilation) operators. It is convenient to use the holomorphic representation of these commutation relations with the following integration measure (the index μ is omitted):

$$1=\int d\omega$$

$$=\prod_{n=1}^{\infty}\int\frac{da_n^*da_n}{2\pi i}e^{-a_n^*a_n}\int dc_n^*db_ndb_n^*dc_ne^{-c_n^*b_n-b_n^*c_n}$$

$$\times\prod_{r=\frac{1}{2}}^{\infty}\int d\psi_r^*d\psi_re^{-\psi_r^*\psi_r}\int\frac{d\gamma_r^*d\gamma_rd\beta_r^*d\beta_r}{(2\pi i)^2}e^{-\beta_r^*\gamma_r+\gamma_r+\gamma_r^*\beta_r}. \quad (3)$$

The Green's function for the operator $\hat{L}_0$ obeys the equation

$$\hat{L}_0G(x,A^*;x',A')=\delta(x,A^*;x',A'). \quad (4)$$

Here x and x' are the positions of the string center of mass, A^* and A' are the sets of all the oscillator variables, and $\delta(x,A^*;x',A')$ stands for the kernel of the unity operator:

$$\delta(x,A^*;x',A')=\delta^{10}(x-x')\sum_k\Delta_k(A^*,A')$$

$$\equiv\delta^{10}(x-x')\sum_k(-)^{k^\gamma}\prod_{n=1}^{\infty}\left\{(c_n^*b_n')^{k_n^c}(b_n^*c_n')^{k_n^b}\prod_{\mu=1}^{10}\frac{(a_{n\mu}^*a_{n\mu}')^{k_{n\mu}^a}}{k_{n\mu}^a!}\right\}$$

$$\times\prod_{r=\frac{1}{2}}^{\infty}\left\{\frac{(\gamma_r^*\beta_r')^{k_r^\gamma}}{k_r^\gamma!}\frac{(\beta_r^*\gamma_r')^{k_r^\beta}}{k_r^\beta!}\prod_{\mu=1}^{10}(\psi_{r\mu}^*\psi_{r\mu}')^{k_{r\mu}^\psi}\right\}, \quad (5)$$

where k is a multi-index which is connected with the occupation numbers. A solution of Eqs. (4) and (5) can be represented in the form

$$G(x,A^*;x',A')=2\sum_k\int\frac{d^{10}p}{(2\pi)^{10}}\frac{e^{ip(x-x')}}{p^2+2[k]-1}\Delta_k(A^*,A'),$$

$$[k]=\sum_{n=1}^{\infty}nk_n+\sum_{r=\frac{1}{2}}^{\infty}rk_r. \quad (6)$$

Actually, this expression is somewhat formal, because there is a singularity connected with the presence of the tachyon in the mass operator spectrum. This is a defect of the theory which will not be discussed here. We note, however, that this difficulty is absent in the R-sector.

Using a summation over the mass operator spectrum, we can reduce this representation to the Källen–Lehmann form

$$G(x,A^*;x',A')=\sum_{M^2}\int\frac{d^{10}p}{(2\pi)^{10}}\frac{e^{ip(x-x')}}{p^2+M^2}\rho(M^2;A^*,A'), \quad (7)$$

where the spectral density matrix is

$$\rho(M^2;A^*,A')=\int_{-1}^{1}d\phi e^{ix\phi(M^2+1)}\sum_k e^{-2i\pi[k]\phi}\Delta_k(A^*,A'). \tag{8}$$

Taking a trace over all the oscillator variables, we obtain the spectral density for the fixed x and x' only:

$$\rho(M^2)=\mathrm{Tr}\rho(M^2;A^*,A')=\int_{-1}^{1}d\phi e^{i\pi\phi(M^2+1)}\prod_{n=1}^{\infty}\left[\frac{1+e^{-i\pi\phi(2n-1)}}{1-e^{-2i\pi\phi n}}\right]^8. \tag{9}$$

The asymptotic behavior of $\rho(M^2)$ is known to be linear exponential.[1,2,8] For more general boundary conditions we can use the method of Ref. 4. If $\rho\equiv\rho(M^2;A^*,A')$ is a compact operator and $R(A^*,A')$ is a bounded operator in the Hilbert space, then $|\mathrm{Tr}\rho|\leqslant\|\rho\|_1$ and $|\mathrm{Tr}R\rho|\leqslant\|\rho\|_1|R\|$, where $\|\rho\|_1$ is the nuclear norm of the operator ρ. It can be shown that in our case the nuclear norm of ρ coincides with $\rho(M^2)$. Therefore the spectral density $\mathrm{Tr}R\rho$ behaves as a linear exponent for any bounded R in the Källen–Lehmann-like representation (7).

If we choose the initial state and the final state in the form

$$\Phi(X^{i,f};A^*)=\prod_{n=1}^{\infty}\left\{\frac{1}{\sqrt{2\pi^5}}\exp(c_n^*b_n^*)\prod_{\mu=1}^{10}\exp\left[-\frac{1}{2}(X_{n\mu}^{i,f})^2-i\sqrt{2}X_{n\mu}^{i,f}a_{n\mu}^*+\frac{1}{2}(a_{n\mu}^*)^2\right]\right\} \tag{10}$$

and take a trace over the other oscillator variables, we obtain the following expression for the spectral density:

$$\rho(M^2;X^f,X^i)=N^{-1}\int_{-1}^{1}d\phi e^{i\pi\phi(M^2+1)}\prod_{n=1}^{\infty}\frac{(1+e^{i\pi\phi(2n-1)})^8}{(1-e^{-4i\pi\phi n})^4}$$
$$\times\exp\left\{i\sum_{n=1}^{\infty}\frac{[(X_n^f)^2+(X_n^i)^2]\cos(2\pi n\phi)-2X_n^fX_n^i}{2\sin(2\pi n\phi)}\right\},\quad N=\prod_{n=1}^{\infty}2\pi^5. \tag{11}$$

This result corresponds to the case in which the initial and final string positions in the space-time are fixed. In this case the Green's function can be written in the form ($\Delta x\equiv x-x'$):

$$G(X^f,X^i)=\frac{1}{32N\pi^9}\int_0^{\infty}\frac{dT}{T^5}e^{\pi T/2}\prod_{n=1}^{\infty}\frac{(1+e^{-\pi T(n-1/2)})^8}{(1-e^{-2\pi nT})^4}$$
$$\times\exp\left\{-\frac{(\Delta x)^2}{2\pi T}-\sum_{n=1}^{\infty}\frac{[(X_n^f)^2+(X_n^i)^2]\cosh(\pi nT)-2X_n^fX_n^i}{2\sinh(\pi nT)}\right\}, \tag{12}$$

which coincides with the result of the path integration.[1,2]

Let us consider the R-sector. The equation for the Green's function can now be written in the form

$$\hat{F}_0G(x,A^*;x',A')=\mathrm{I}\delta(x,A^*;x',A'), \tag{13}$$

where

$$\hat{F}_0=\frac{1}{i\sqrt{2}}\hat{p}\Gamma+\Gamma_{11}\left[\sum_{n=1}^{\infty}\left\{\sqrt{n}(\hat{a}_{-n}\hat{\psi}_n+x\hat{\psi}_{-n}\hat{a}_n)-2(\hat{b}_{-n}\hat{\gamma}_n+\hat{\gamma}_{-n}\hat{b}_n)\right.\right.$$

$$\left.\left.-\frac{1}{2}n(\hat{\beta}_{-n}\hat{c}_n-\hat{c}_{-n}\hat{\beta}_n)\right\}-2\gamma_0b_0\right], \tag{14}$$

and I is a unit matrix. The solution of (13) can be represented in the form

$$G(x,A^*;x',A')=\sum_{M^2}\int\frac{d^{10}p}{(2\pi)^{10}}\frac{e^{ip(x-x\prime)}}{p^2+M^2}[p\Gamma\rho_1(M^2;A^*,A')$$

$$+\Gamma_{11}\rho_2(M^2;A^*,A')], \tag{15}$$

where the spectral densities are

$$\rho_1(M^2;A^*,A')=-\frac{i}{\sqrt{2}}\int_{-1}^{1}d\phi e^{i\pi\phi M^2}\sum_k e^{-2i\pi[k]\phi}\Delta_k(A^*,A'),$$

$$\rho_2(M^2;A^*,A')=\int_{-1}^{1}d\phi e^{i\pi\phi M^2}\sum_{m=1}^{\infty}e^{-2i\pi m\phi}\left\{\sqrt{m}(a_m^*\psi_m'+\psi_m^*a_m')\right.$$

$$\left.-2(b_m^*\gamma_m'+\gamma_m^*b_m')-\frac{1}{2}m(\beta_m^*c_m'-c_m^*\beta_m')\right\}$$

$$\times\sum_k e^{-2i\pi[k]\phi}\Delta_k(A^*,A'). \tag{16}$$

Taking a trace over all the oscillator variables, we obtain

$$\rho_1(M^2)=\mathrm{Tr}\rho_1(M^2;A^*,A')=-\frac{i}{\sqrt{2}}\int_{-1}^{1}d\phi\epsilon^{i\pi\phi M^2}\prod_{n=1}^{\infty}\left[\frac{1+\epsilon^{-2i\pi\phi n}}{1-\epsilon^{-2i\pi\phi n}}\right]^8,$$

$$\rho_2(M^2)=\mathrm{Tr}\rho_2(M^2;A^*,A')=0. \tag{17}$$

Choosing the boundary conditions (10), we obtain

$$\rho_2(M^2;X^f,X^i)=0,$$

$$\rho_1(M^2;X^f,X^i)=-\frac{i}{\sqrt{2}}N^{-1}\int_{-1}^{1}d\phi e^{i\pi\phi M^2}\prod_{n=1}^{\infty}\left(\frac{1+e^{-2i\pi\phi n}}{1-e^{-2i\pi\phi n}}\right)^4$$

$$\times\exp\left\{i\sum_{n=1}^{\infty}\frac{[(X_n^f)^2+(X_n^i)^2]\cos(2\pi n\phi)-2X_n^fX_n^i}{2\sin(2\pi n\phi)}\right\},$$

$$N=\prod_{n=1}^{\infty}2\pi^5; \tag{18}$$

$$G(X^f,X^i)=-\frac{1}{32\sqrt{2N\pi^9}}\left(\Gamma\frac{\partial}{\partial x}\right)\int_0^\infty\frac{dT}{T^5}\prod_{n=1}^\infty\left[\frac{1+e^{-\pi nT}}{1-e^{-\pi nT}}\right]^4$$
$$\times\exp\left\{-\frac{(\Delta x)^2}{2\pi T}-\sum_{n=1}^\infty\frac{((X_n^f)^2+(X_n^i)^2)\cosh(\pi nT)-2X_n^fX_n^i}{2\sinh(\pi nT)}\right\}. \quad (19)$$

Equation (19) coincides with the path integration result.[1,2]

Using the results obtained for the NS-sector, we can show that the growth of the spectral densities is linear exponential for any boundary conditions.

Let us discuss the singular properties of the propagators (12) and (19). In addition to the infrared tachyon singularity in the NS-sector, these two expressions possess the ultraviolet ($T\to 0$) singularities which are connected with the exponential growth of the spectral densities. Taking into account the properties of the θ-functions (see, for example, Ref. 8), we obtain the equations

$$e^{\pi T/2}\prod_{n=1}^\infty\frac{(1+e^{-\pi T(n-1/2)})^8}{(1-e^{-2\pi nT})^4}=T^2e^{\pi/T}\prod_{n=1}^\infty\frac{(1+e^{-\frac{2\pi(2n-1)}{T}})^8}{(1-e^{-\frac{2\pi n}{T}})^4} \quad (20)$$

for the NS-sector and

$$\prod_{n=1}^\infty\frac{(1+e^{-\pi nT})^8}{(1-e^{-2\pi nT})^4}=\frac{T^2}{16}e^{\pi/T}\prod_{n=1}^\infty\frac{(1-e^{-\frac{2\pi(2n-1)}{T}})^8}{(1-e^{-\frac{2\pi n}{T}})^4} \quad (21)$$

for the R-sector. It can be easily seen that the main ultraviolet singularity is proportional to

$$\int_0\frac{dT}{T^3}\exp\left[\frac{\pi}{T}-\frac{(\Delta x)^2}{4\pi\alpha' T}\right]\quad\text{for the NS-sector} \quad (22)$$

and proportional to

$$\frac{\partial}{\partial x}\int_0\frac{dT}{T^3}\exp\left[\frac{\pi}{T}-\frac{(\Delta x)^2}{4\pi\alpha' T}\right]\quad\text{for the R-sector.} \quad (23)$$

This confirms the general conclusion obtained in Ref. 1 [see Eq. (4.56)].

To study the causal properties of the string propagator, it is necessary to transform to the Minkowski space-time by means of the analytic continuation in p_0. Strictly speaking, this can be done if we ignore the tachyon state in the NS-sector. We then see from (7), (15), (22), and (23) that the propagators (12) and (19) have noncasual singularities in the space-like region $(\Delta x)^2=(\Delta x)^2-(\Delta x_0)^2\geqslant 4\pi^2\alpha'$. This region does not increase if more general boundary conditions are chosen.

For the closed fermionic string the Källen–Lehmann-like representation can also be derived. Equations like (7) and (12) for the NS–NS-sector, like (15), (16), and (19) for the R–NS-sector, and more complicated equations for the R–R sector will be given in another paper.

In conclusion, we would like to note that the question of the minimal region of nonlocality in string theory is closely connected with the remarkable role of the modular invariance in the vanishing of the ultraviolet divergences. We do not yet know how to join the ends of the closed string propagator in order to obtain the fundamental region of the integration over modular parameters automatically.[8] This is a task of a future field theory of closed strings.

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[1]V. Ya. Fainberg and A. V. Marshakov, Phys. Lett. B **211**, 82 (1988); Proceedings of the Lebedev Physics Institute **201**, 139–179 (1990).
[2]A. V. Marshakov, Nucl. Phys. B **312**, 178 (1989).
[3]A. M. Polyakov, Phys. Lett. B **103**, 207,211 (1981).
[4]V. Ya. Fainberg and M. A. Soloviev, Theor. Math. Phys. **93**, 1438 (1992).
[5]S. Fubini and G. Veneziano, Nuovo Cim. A **64**, 811 (1969); K. Bordakei and S. Mandelstam, Phys. Rev. **184**, 1640 (1969).
[6]M. Z. Iofa and V. Ya. Fainberg, Sov. Phys. JETP **29**, 880 (1969); Theor. Math. Phys. **1**, 143 (1969).
[7]V. Ya. Fainberg and M. A. Soloviev, Ann. of Phys. **113**, 421 (1978),
[8]M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (1987), Cambridge Univ. Press, Cambridge.

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