Is the operator expansion series in deep inelastic lepton–hadron scattering asymptotic or converging?

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This question is studied by analyzing the scattering of electrons by a system of two particles coupled by an oscillator potential in nonrelativistic quantum mechanics. The series is shown to be asymptotic in this problem. It is suggested that this property of the series is characteristic of theories with confinement.

The question of whether the operator expansion series in deep inelastic lepton–hadron scattering is asymptotic or converging surprisingly has not been discussed in the literature, to the best of my knowledge. Clearly, a resolution of this question would be of both theoretical and practical value. It would be of practical value because a better understanding of the nature of the convergence of the series in inverse powers of the square of the momentum transfer, $1/q^2$, in deep inelastic lepton–hadron scattering would be of assistance in evaluating the range of applicability of perturbative QCD in these processes.

One can cite heuristic arguments in favor of both possible answers to the question posed in the title of this letter. A first argument is based on the circumstance that the point $q^2=\infty$, near which the operator expansion is carried out in momentum space, is an accumulation point of singularities corresponding to the thresholds for various channels. One might thus expect that the series expansion would be asymptotic near this point. That argument is not completely convincing, since an accumulation of singularities at a certain point does not necessarily mean that a power series near this point would be asymptotic. An argument in favor of the opposite conclusion arises when we look at the deep inelastic scattering by a free particle or by a particle which is interacting with a weak external field. It can be shown that the operator expansion series is a converging series in this case.

Below we discuss a model in which this question can be resolved. In nonrelativistic quantum mechanics we consider the scattering of an electron by a system of two particles (a "hadron") which are bound by an oscillator potential (see Ref. 1 for a discussion of deep inelastic scattering in nonrelativistic quantum mechanics and for a definition of the structure function.) We assume for simplicity that the masses m of the particles are equal, that only one of the particles is charged, and that its charge is 1. In the nonrelativistic theory, among the tensor components $W_{\mu\nu}$ which determine the imaginary part of the amplitude for the forward scattering of a virtual photon by a hadron and the structure functions, we are left with only the component W_{00}. Since the wave functions in an oscillator potential are well known and fairly simple, it is a straightforward matter to calculate the structure function in this model. The corresponding calculation was recently carried out by Greenberg.[2] In general, W_{00} is a function of two variables: the square of the momentum transfer from the electron to the hadron, q^2 ($\mathbf{q}$ is a 3-vector in the nonrelativistic theory), and the energy transfer q_0. It is convenient to replace the latter variable by the (Bjorken) scaling variable

$$x=q^2/4mq_0, \quad 0\leqslant x\leqslant 1. \tag{1}$$

For W_{00} Greenberg has found

$$W_{00}(q^2,x)=\frac{1}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{q^2}{4m\omega}\right)^n \exp\left(-\frac{q^2}{4m\omega}\right)\delta\left[\frac{q^2}{4m}\left(\frac{1}{x}-1\right)-n\omega\right], \tag{2}$$

where ω is the oscillator frequency.

Let us assume that $q^2/4m\omega\gg 1$, and that therefore $n\gg 1$ in (2) for values of x not close to one. The nth oscillator level can be excited only if q^2 and x satisfy a certain relation which follows from the vanishing of the argument of the δ-function in (2) and if, on the contrary, each such relation corresponds to the excitation of one definite level of the oscillator. In this regard, deep inelastic scattering in our model is not similar to deep inelastic electron–hadron scattering. In order to achieve such a similarity, we take an average of (2) over q^2 in the Δq^2 interval satisfying the conditions

$$\Delta q^2\ll q^2, \quad \Delta q^2/4m\omega\gg 1. \tag{3}$$

After taking this average using Stirling's formula for $n!$ at large values of n, we find from (2)

$$\bar{W}_{00}(q^2,x)=\frac{1}{(2\pi)^2}\sqrt{\frac{1-x}{kx}}\exp\left[-kf(x)\right], \tag{4}$$

where $k^2=q^2/4m\omega$ and

$$f(x)=\frac{1-x}{x}\ln\frac{1-x}{x}+2-\frac{1}{x}. \tag{5}$$

The function $f(x)$ is nonnegative in the interval $0\leqslant x\leqslant 1$ and vanishes along with its derivative at $x=1/2$. At large values of $k\gg 1$, a value of $\bar{W}_{00}(q^2,x)$ which is not exponentially small arises exclusively because of x values close to 1/2. This case corresponds to an assertion of the parton model, that in the case at hand each of the two partons making up a hadron carries half the momentum of the hadron.

The latter assertion applies only to the leading term in the $1/q^2$ expansion. To study the entire series in $1/q^2$, which is our purpose in the present letter, we consider the moments of the structure function, in particular, the first moment

$$M_1(q^2)\equiv\int_0^1\bar{W}(q^2,x)dx=\frac{1}{(2\pi)^2}\int_0^1 dx\sqrt{\frac{1-x}{kx}}\,e^{-kf(x)}. \tag{6}$$

We will show that expansion (6) in powers of $1/k$ is an asymptotic series. It is convenient to change the integration variable on the right side of (6), setting

$$x=\frac{1}{2}(1+y). \tag{7}$$

At large values of k, small values of y are important on the right side of (6), and we can use a Taylor-series expansion of $f(y)$:

$$f(y)=2y^2\left[1-\frac{4}{3}y-\frac{1}{3}y^2-\sum_{n=1}^{\infty} y^{2n+1}\left(1+y+\frac{1-2y}{2n+3}\right)\right]. \quad (8)$$

After (8) is substituted into (6), the first term on the right side of (8) gives us a scaling $M_1(k)\sim 1/k$. The contribution of the terms of higher order in y in (8) corresponds to terms of higher twist in deep inelastic scattering. Since all the terms of higher order in y are in the exponential function $e^{-kf(y)}$, with the same positive sign, they cannot cancel out. Accordingly, in order to prove that expansion (6) in powers of $1/k$ is asymptotic, it is sufficient to expand the exponential function in (6) (other than the first term e^{-2ky^2}) in power of y and to select a sequence of terms in this expansion which would lead to an asymptotic series. The terms discarded in the course of this procedure could only degrade the convergence of the asymptotic series. As this sequence of terms we choose the first term in the expansion of the exponential function:

$$\exp\left(2ky^2\sum_{n=3}^{\infty} y^n\right)\approx 1+2ky^2\sum_{n=3}^{\infty} y^n. \quad (9)$$

Substitution of (9) into (6) leads to an asymptotic series in $1/k$ for $M_1(k)$ with higher-order terms proportional to

$$\frac{1}{2^{n+1}}\frac{1}{k^{(n+1)/2}}\Gamma\left(\frac{n+3}{2}\right). \quad (10)$$

Since the expansion of the exponential function in (6) in powers of y incorporated only some of the terms, the convergence of the entire asymptotic series might be considerably worse.

One step in the proof given above might raise some eyebrows: the use of Stirling's formula for $n!$ at large n in the derivation of Eq. (4). The terms of higher order in $1/n$, which were discarded in the derivation of (4), lead to terms of higher order in $1/q^2$. The latter might in principle compensate in a factorial way for the increasing coefficients in asymptotic series (10) and might lead to a converging series, although that possibility looks very implausible.

The possibility of such a compensation could be rejected on the basis of physical grounds. The procedure of taking an average over q^2, which was used in establishing the correspondence between this model and deep inelastic scattering by hadrons, is well defined only in the limit of large q^2, in which inequalities (3) hold. When terms of higher order in $1/q^2$ are taken into account, this procedure becomes ambiguous. Since expression (4) for $\bar{W}_{00}(q^2,x)$ depends on the averaging procedure in higher orders in $1/q^2$, the terms which appear as corrections to Stirling's formula cannot, in general, cancel asymptotic series (10), which does not depend on the averaging procedure. Our assertion thus remains valid when that circumstance is taken into account.

Clearly, the arguments presented above can be repeated for any moment of the structure function, and the conclusion will be qualitatively the same. We reach the conclusion that in the problem discussed above–the scattering of an electron by a system of two particles which are bound by an oscillator potential–the operator expansion series is an asymptotic series in $1/q^2$. The model discussed here is extremely specific. It is a model with confinement; specifically, the particles which do the scattering cannot be observed in a free state. In this regard, the model is somewhat similar to QCD. On the other hand, as was mentioned above, in the problem of deep inelastic scattering by a free particle or by a particle moving in a weak external field we would expect that an operator expansion series in $1/q^2$ would converge. We can thus offer the hypothesis that the asymptotic nature of the operator expansion series is a characteristic property of confinement theories which are similar to QCD. It would be extremely interesting to see a test of this hypothesis on other models.

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