Moments of multiplicity distributions in higher-order perturbation theory in QCD

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The ratio of the cumulant moments of the multiplicity distributions of partons to the factorial moments exhibits some qualitatively distinct features (minima and maxima) when higher-order terms of the expansion of perturbative QCD are incorporated in the nonlinear equation for the generating function. The correspondence between these results and experimental data is discussed.

Describing the multiplicity distribution of particles produced in high-energy inelastic processes remains one of the most important problems in quantum chromodynamics (QCD). In the lowest-order log-log approximation the theory predicts a distribution much broader than those which have been observed experimentally[1] in e^+e^- collisions. It was recently shown[2,3] that the theoretical distribution becomes narrower if the nonlinearity of the equations for the generating functions is taken into account correctly in QCD. A new characteristic of the distributions, $H_q = K_q/F_q$, was proposed in Ref. 3. This is the ratio of the cumulant moments of these distributions to the factorial moments. It turns out to be extremely sensitive to subtle details of the distributions and to small variations of the factorial moments. A minimum of this ratio has been predicted[3] at $q \approx 5$, and this minimum has been verified experimentally.[4] The experimental data have also revealed oscillations in H_q at large values of q, while it was concluded in Ref. 3 that H_q becomes asymptotically constant when only the lowest-order terms of the Taylor-series expansion of the generating function are retained in the corresponding nonlinear equation.

We show below that this difference in qualitative features of the function H_q disappears when higher-order terms of the expansion are taken into account. These terms also give rise to oscillations. The reader can see the idea quite easily by imagining how the oscillations of an ordinary cosine would be reproduced progressively more accurately at progressively larger values of its argument through a systematic account of higher-order terms in the Taylor-series expansion. We restrict the discussion in this letter to the case of gluodynamics, ignoring quarks (which play a relatively minor role[5]). We make no claim, therefore, that we are attempting a quantitative description of experiments (see also Ref. 5). Our approach is to estimate the nonlinearity in the equation for the generating function $G(z)$ more accurately (than in Refs. 1 and 3). In gluodynamics, this equation is

$$G'(y,z) = \int_0^1 d\xi [\xi^{-1} - \Phi_r(\xi)] \gamma_0^2 [G[y+\ln(1-\xi),z] G(y+\ln\xi,z) - G(y,z)], \tag{1}$$

where $\gamma_0^2=2N_c\alpha_s/\pi$, α_s is a running coupling constant, $N_c=3$ is the number of colors, $\Phi_r(\xi)=(1-\xi)[2-\xi(1-\xi)]$ is the regular part of the kernel, the prime means the derivative with respect to $y=\ln Q/Q_0$ (Q_0=const), and Q is a large characteristic transverse momentum of the gluon jet.

The generating function is defined by

$$G(y,z)=\sum_{n=0}^{\infty} P_n(y)(1+z)^n, \tag{2}$$

where $P_n(y)$ is the distribution of partons in the gluon jet. The factorial and cumulant moments are related to $G(z)$ by

$$G(y,z)=1+\sum_{q=1}^{\infty}\frac{z^q}{q!}F_q\langle n(y)\rangle^q, \tag{3}$$

$$\ln G(z)=\sum_{q=1}^{\infty}\frac{z^q}{q!}K_q\langle n(y)\rangle^q, \tag{4}$$

where $\langle n\rangle$ is the mean multiplicity of the partons in a jet with a given y.

Expanding G in a Taylor series around the point y, and carrying out some straightforward calculations, we find from (1)

$$G'(y)=G(y)\left[\int_{-\infty}^{y} dy'\gamma_0^2(y')[G(y')-1]-2h_1\gamma_0^2[G(y)-1]\right.$$
$$\left.+\gamma_0^2\sum_{n=1}^{\infty} I_n G^{(n)}(y)\right]+\gamma_0^2\sum_{m,n}^{\infty} I_{mn}G^{(m)}(y)G^{(n)}(y), \tag{5}$$

where $h_1=11/24$,

$$I_{mn}=\frac{1}{n!\,m!}\int_0^1 d\xi[\xi^{-1}-\Phi_r(\xi)]\ln^n(\xi)\ln^m(1-\xi),$$

$$I_n=\frac{1}{n!}\int_0^1 d\xi[\xi^{-1}-2\Phi_r(\xi)]\ln^n(1-\xi)$$
$$=(-1)^{n+1}[2-2^{-n-1}-3^{-n-1}-\zeta(n+1)],$$

and $\zeta(n)=\Sigma_{m=1}^{\infty} m^{-n}$ is the Riemann zeta function.

Dividing both sides of Eq. (5) by G, and differentiating with respect to y, we find the equation

$$[\ln G(y)]''=\gamma_0^2\left[G(y)-1-2h_1G'(y)+\sum_{n=2}^{\infty}(-1)^n h_n G^{(n)}(y)\right.$$
$$\left.+\sum_{m,n=1}^{\infty}(-1)^{n+m}h_{nm}\left(\frac{G^{(m)}G^{(n)}}{G}\right)'\right], \tag{6}$$

where $h_n=|I_{n-1}|$ for $n\geqslant 2$, and $h_{mn}=|I_{mn}|$. In going over from our original equation, (1), to (6) we ignored only terms of the type $d\gamma_0^2(y)/dy\sim\gamma_0^4(y)$.

Let us examine the expansion parameter in Eq. (6). For this purpose we note that only the quantity $\langle n \rangle$ depends on y in Eqs. (3) and (4), since F_q and K_q are constant over y by virtue of KNO scaling. Differentiating it with respect to y, we find

$$\frac{d^n}{dy^n}\langle n\rangle^q=(\gamma q)^n\left(1+\frac{n(n-1)}{2}\frac{1}{\gamma q}\frac{\gamma'}{\gamma}+...\right)\langle n\rangle^q, \tag{7}$$

where $\langle n\rangle=\exp(\int^y\gamma(y')dy')$, and γ is the anomalous QCD dimensionality. With $\gamma=\gamma_0=\text{const}$ we find $\langle n\rangle=\exp(\gamma_0 y)$ and $d^n\langle n\rangle^q/dy^n=(\gamma_0 q)^n\langle n\rangle^q$. The quantity $x\equiv\gamma q$, not the perturbation-theory parameter γ, is the actual expansion parameter, as was mentioned in Ref. 1 (this point was actually utilized in Refs. 2 and 3). Expressions for γ and γ' in terms of γ_0 in higher orders are given in Ref. 3. We recall, in particular, that we have $\gamma=\gamma_0+O(\gamma_0^2)$ and $\gamma'=-h_1\gamma_0^3+O(\gamma_0^4)$.

Equation (6) was solved in Ref. 3 to within terms on the order of x^2. The terms with $N>2$ in the first sum and the entire second sum were discarded. The solution leads to H_q with a single minimum, at $q\approx 5$, and with an asymptotically constant value $\gamma_0^2 h_2$ as $q\to\infty$ (Ref. 3).

We now take account of terms of the next higher order in Eq. (6). The latter becomes

$$[\ln G]''=\gamma_0^2[G-1-2h_1G'+h_2G''-h_3G'''+h_{11}(G'(\ln G)')'] \tag{8}$$

or, after we use (3) and (4),

$$(q^2\gamma^2+q\gamma')K_q=\gamma_0^2\Big\{F_q[1-2h_1q\gamma+h_2(q^2\gamma^2+q\gamma')-h_3(q^3\gamma^3+3q^2\gamma\gamma'+q\gamma'')]$$
$$+h_{11}\sum_{k=1}^{q-1}C_q^kK_{q-k}F_k(q-k)k\gamma(q\gamma^2+2\gamma')\Big\}, \tag{9}$$

where $C_q^k=q!/k!\,(q-k)!$ are the binomial coefficients. The coefficient h_{11} can be calculated very accurately by writing it in series form:

$$h_{11}=\frac{7}{8}+\frac{8}{27}-\sum_{n=4}^{\infty}\frac{1}{n^2}\left[\frac{1}{n(n-1)}+\frac{1}{2(n-2)(n-3)}\right]\approx 0.8812.$$

For convenience, we introduce the notation $k_q=K_q/(q-1)!$ and $f_q=F_q/(q-1)!$ The cumulant and factorial moments are related by a well-known equation, which takes the following form in terms of our new notation (k_q and f_q):

$$f_q=k_q+\sum_{k=1}^{q-1}k^{-1}f_kk_{q-k}. \tag{10}$$

Discarding only the terms of order x^3 in (9), and using (10), we find yet another relation between k_q and f_q:

$$k_q=\frac{1}{1-H_q^{(3)}}\sum_{k=1}^{q-1}f_kk_{q-k}\left[\frac{H_q^{(3)}}{k}-\gamma_0^2\left(\frac{h_3}{k}-\frac{h_{11}}{q}\right)x\right], \tag{11}$$

where

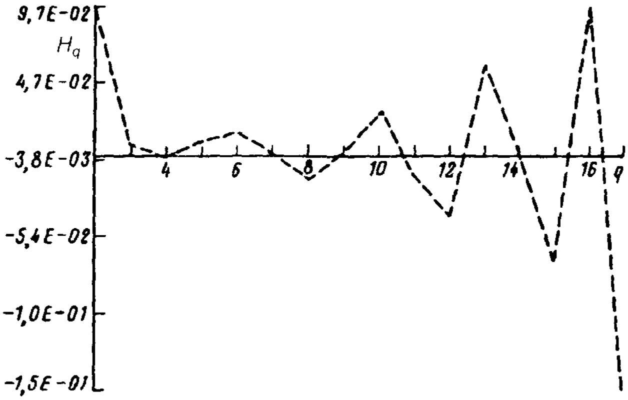

FIG. 1.

$$H_q^{(3)}=\gamma_0^2\left(h_2-2h_3\frac{\gamma'}{\gamma}-\frac{2h_1}{x}+\frac{1}{x^2}\right).$$

Equations (10) and (11) thus unambiguously determine the behavior of the two functions k_q and f_q and thus the behavior of their ratio H_q. We have found the function H_q by numerical calculations; the results are shown in Fig. 1.

A characteristic feature of the ratio H_q in this approximation, and a feature which distinguishes this ratio from that in a lower approximation,[3] is the oscillatory behavior at large values of q, instead of the tendency toward a constant asymptote. We should point out that the first minimum is shifted toward $q=4$. The oscillation amplitude increases with increasing q, while the oscillation period decreases to the extent that at $q\geqslant 14$ we begin to see a change in the sign of H_q at each successive value of q.

The qualitative picture, with oscillations of the function H_q, which was first noted in an analysis of experimental data,[4] is thus reproduced in this order of the QCD expansion. In addition, we find a prediction that the sign of H_q changes upon each unit increment in q at large values of q. This prediction can be explained by analogy with the behavior of H_q in events with a fixed multiplicity,[6] since the main peak in the P_n distribution at large values of q "looks" infinitely narrow; this circumstance leads to a change in the signs of the cumulants.

We wish to stress a difference between Eqs. (10) and (11). While the first gives a purely mathematical relationship between the moments, the second is determined by the dynamics of the process, in the case at hand by the QCD equations. The latter equations also lead to oscillations, in contrast with, say, a negative binomial distribution, whose moments do not oscillate.[6] The period and amplitude of the oscillations observed experimentally may point out ways for modifying the equations for the generating functions in QCD.

The effect observed in the present paper turns out to be very sensitive to small perturbations. For example, the results change substantially if we set $\gamma'=0$, i.e., if we assume that the coupling constant is not a running constant. Although the oscillations at comparatively small values of q remain, in a slightly different form, the solution "cuts off" at large values $q=15$, leading to negative values of the factorial moments, in contradiction of their definition. A similar cutoff can be seen at much lower values, $q=5$, in a previous study,[7] in which a different approximation was used to solve Eq. (1). This result emphasizes just how accurate and systematic one must be in dealing with terms of the same order of magnitude in this equation. It is extremely important to note that the magnitude and qualitative behavior of H_q turn out to be very sensitive to the slightest changes in F_q, even at comparatively small values of q, which are difficult to distinguish in the standard approach. The function H_q should be used to bring out subtle features in multiplicity distributions.

We end on a note of caution, regarding attempts to compare Fig. 1 directly with experimental data. Such a comparison is not possible at a quantitative level, since the numerical values may be altered by the incorporation of quarks, higher-order terms of the expansion, and possibly confinement. On the other hand, we believe that the very fact that a "quasi-oscillatory" behavior of the function H_q arises in higher orders of the theory deserves attention and further study.

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