Topology of vortex–soliton intersection: Invariants and Torus homotopy

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(Submitted 24 November 1994)
Pis'ma Zh. Eksp. Teor. Fiz. **61**, No. 1, 48–53 (10 January 1995)

The topology relevant to the intersection of nonsingular 4π vortex lines with a planar transverse soliton in superfluid ^{3}He-A, which was recently observed in Helsinki, is discussed. The essential part of the problem consists in finding a homotopy classification of mappings $S^2\times S^1\rightarrow S^2$ and $S^1\times S^1\times S^1\rightarrow S^2$. This classification is achieved, and an analytical expression for the topological invariant is found. This expression is analogous to that for the Hopf invariant $H=(1/4\kappa^2)\int \mathbf{v}_s \text{curl}\, \mathbf{v}_s g V$ of mappings $S^3\rightarrow S^2$. © *1995 American Institute of Physics.*

Many different types of topological defects have been suggested in condensed matter with broken symmetry. A number of them have been experimentally discovered by now. A perfect mathematical tool for their investigation is given by homotopy theory which has been extensively used in physics.[1,2] In many important cases it is sufficient to know the order parameter space and be able to calculate its homotopy groups π_n. In particular, knowledge of homotopy groups makes it possible to classify possible free monopoles, vortices, and interfaces, defects which appear ubiquitous in the universe.[2,3] Classification of defects which are caused by some sort of boundary conditions or which are confined to other defects is more laborious, leading in a simple case to relative homotopy groups.[4,5]

Experimental evidence for the coexistence of a planar soliton, which is pierced with nonsingular 4π vortices in a rotating ^{3}He-A, was recently reported.[6] The purpose of this article is to extend the discussion of the underlying topology, including the analytical expressions for topological invariants. We will show that the essential part of the problem consists in topological classification of mappings, $S^2\times S^1\rightarrow S^2$ and $S^1\times S^1\times S^1\rightarrow S^2$. The latter set of mappings is sometimes called "torus homotopy group" of 2-sphere $T^3(S^2)$ and is alleged to be unknown,[7] although it actually was first found by Pontrjagin[8] in 1941.

Geometry of the problem. Let $\hat{z}$ be the axis of rotation of the vessel which contains ^{3}He-A. The vector $\mathbf{d}$ of the magnetic anisotropy is confined by the magnetic field to the horizontal plane $\mathbf{d}\perp\hat{z}$. Furthermore, we can safely assume that $\mathbf{d}=\mathbf{d}_0$ is everywhere constant. This reduces the order parameter to its orbital part $\Psi=\mathbf{e}_1+i\mathbf{e}_2$, where $\mathbf{e}_1\perp\mathbf{e}_2$, and $\mathbf{e}_1^2=\mathbf{e}_2^2=1$. This simplification does not alter the topological classification, because the $\mathbf{d}$ vector has no net winding either inside the soliton or inside vortices under consideration.[6]

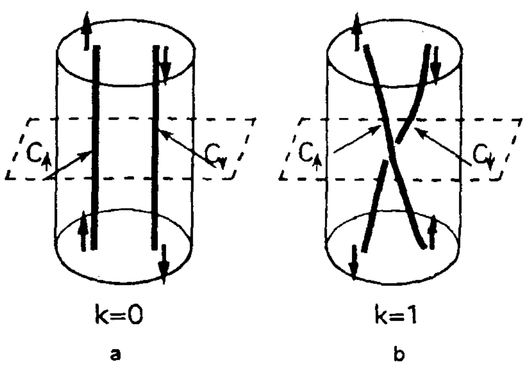

FIG. 1. Topology of two different structures of a vortex crossing soliton (before duplication). a—Loci $C_\uparrow$ and $C_\downarrow$ at which $\mathbf{l}=\pm\hat{z}$ do not intersect; b—where they intersect.

The remaining manifold of degenerate states is thus SO_3. A small spin–orbit interaction tends to fix the vector of the orbital anisotropy $\mathbf{l}=\mathbf{e}_1\times\mathbf{e}_2$ to $\pm\ \mathbf{d}_0$, which restricts the degeneracy space to $S^1\times Z_2$. Nontriviality of the relative homotopy groups $\pi_2(SO_3;S^1\times Z_2)=Z$ and $\pi_1(SO_3;S^1\times Z_2)=Z_2$ gives rise to the possibility of nonsingular 4π vortices and solitons.[4,9] In the geometry of the Helsinki experiment the magnetic field and the vortices are oriented along $\hat{z}$, while the soliton plane is perpendicular to $\hat{z}$, which leads to a particular type of soliton called a twist soliton.[10] The vector $\mathbf{l}$ rotates from $\mathbf{d}_0$ to $-\mathbf{d}_0$ inside the soliton:

$$\mathbf{l}_{\rm sol}=\mathbf{U}_z\mathbf{d}_0\ , \tag{1}$$

where $\mathbf{U}_z$ is the matrix of rotation in the xy plane by an angle $\alpha(z)$, which increases from 0 to π. The whole sphere is swept by the vector $\mathbf{l}$ in any horizontal cross section of a 4π vortex, while far from the vortex $\mathbf{l}$ becomes constant. A possible ansatz is as follows:

$$\mathbf{l}=\mathbf{U}_z(\hat{y}\ \sin\ \eta(r)+\cos\ \eta(r)[\hat{x}\ \sin(\varphi-\varphi_0(z))-\hat{z}\ \cos(\varphi-\varphi_0(z))]), \tag{2}$$

where r,φ are the polar coordinates in the xy plane; $\eta(r)$ varies from $\eta(0)=-\pi/2$ to $\eta(\infty)=\pi/2$, $\varphi_0(z)$ is a function of z, and we have chosen $\mathbf{d}_0=\hat{y}$.

Consider the loci $C_\uparrow$ and $C_\downarrow$ of the points at which the vector $\mathbf{l}$ equals $\pm\hat{z}$, respectively. Depending on the function $\varphi_0(z)$, they interlace or do not interlace (Fig. 1). This notion of interlacing has a topological sense, and the distinction between these two possibilities can in principle be resolved in the experiment. Here we have implicitly assumed that the values $\mathbf{l}(r)=\pm\hat{z}$ are in the generic position, so that $C_\uparrow$ and $C_\downarrow$ are closed curves, possibly consisting of several disjoint links. The same assumption is made below in considering the loci $C_{\mathbf{l}_0}$ of the points $\mathbf{r}$, at which $\mathbf{l}(\mathbf{r})=\mathbf{l}_0$ with an arbitrary $\mathbf{l}_0$.

Formal description: low rotational velocities. Following,[6] we duplicate the soliton, which gives us periodic boundary conditions in the z direction, and take the region of the intersection by a large cylinder (Fig. 2a). If the density of vortices in the vortex lattice is small, which corresponds to low rotational velocities, then each individual vortex is well-defined, permitting us to impose constant boundary conditions at the boundary ∂D^2 of any cross section D^2 of the cylinder. With constant boundary conditions, each cross section effectively becomes a 2-sphere S^2, and the whole volume inside the cylinder should be thought of as a direct product $S^2\times S^1$ of this sphere and the $\hat{z}$-axis

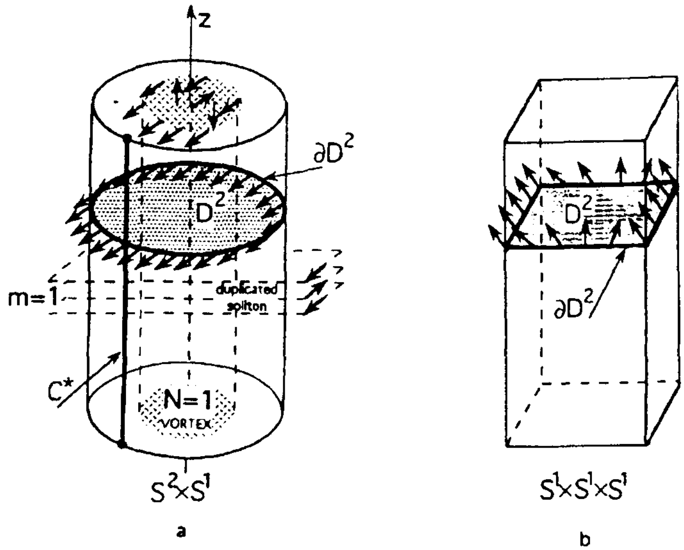

FIG. 2. a—The region of the vortex–soliton intersection by a cylinder. Periodic boundary conditions are imposed along z. Sketched is the case with $N=1$, $m=1$. C^* is an arbitrary vertical generatrix of the lateral surface. For a low density of vortices **l** is constant at each boundary circumference ∂D^2; therefore, the cylinder effectively becomes $S^2\times S^1$. b—For higher densities the boundary conditions on ∂D^2 are periodic, and the cylinder becomes $S^1\times S^1\times S^1$. The case with zero additional invariants m_1, m_2 is shown.

circumference. In addition, the vector **l** on the lateral surface of the cylinder is confined to the xy plane, as it should be for the soliton structure, which is restored far from the vortex. As the density of vortices increases, the vector **l** starts to deviate from the xy plane. Finally, at large densities (large rotational velocities) the intervortex distance and the size of an individual vortex become of the same order of magnitude. We can no longer assume constant boundary conditions in the cross sections, but rather periodic ones. Each cross section of the cylinder effectively becomes a torus $S^1\times S^1$, and the whole cylinder will be $S^1\times S^1\times S^1$. Note that of the three degrees of freedom of $\Psi=\mathbf{e}_1+i\mathbf{e}_2$ we are considering only two carried by **l**. We must do this, because periodic boundary conditions are imposed only on **l** (Ref. 11), and one can, in fact, show that the third component does not alter the topological classification.

Now let us start the classification of possible structures. We express the invariants in terms of the superfluid velocity $\mathbf{v}_s=(\kappa/2\pi)\mathbf{e}_1\nabla\mathbf{e}^2$, which is related to the vector **l** by the Mermin–Ho relation[12]

$$\mathrm{curl}\,\mathbf{v}_s=(\kappa/4\pi)\epsilon_{ijk}l_i\nabla l_j\times\nabla l_k. \qquad (3)$$

Here $\kappa=\pi\hbar/m_3$ is the circulation quantum for superfluid ^{3}He. First, consider the case of **l** confined to the xy plane at the boundary of the cell. This case corresponds to the classification of all mappings of pairs $(D^2\times S^1;\ \partial D^2\times S^1)\rightarrow(S^2;S^1)$, i.e., the mappings $D^2\times S^1\rightarrow S^2$, $\mathbf{r}\mapsto\mathbf{l}(\mathbf{r})$ which map the boundary $\partial D^2\times S^1$ of the cell to the equator S^1 of the **l** sphere. We restrict the discussion to the case in which the boundary ∂D^2 of a cross section of the cell is not wound over the equator by the mapping $\mathbf{l}(\mathbf{r})$ (this means that the corresponding mapping $\partial D^2 \to S^1$ is topologically trivial). This is the case for both constant (S^2) and periodic ($S^1 \times S^1$) boundary conditions. In this case the distribution of the vector $\mathbf{l}$ in the cell is described by three integer topological invariants:

1) The number N of 4π vortices inside the cylinder (usually $N=1$), given by

$$N=\frac{1}{2\kappa}\oint_{\partial D^2}\mathbf{v}_s d\mathbf{r}, \tag{4}$$

where the integral is taken along the boundary of the horizontal cross section.

2) The algebraic number m of solitons before duplication (i.e., the number of solitons with a given direction of twist minus the number of solitons with the opposite direction of twist) that shows how many times the vector $\mathbf{l}$ goes around the equator of the sphere as $\mathbf{r}$ goes along any vertical generatrix of the lateral surface of the cylinder.

3) The linking number k of $C_\uparrow$ and $C_\downarrow$.

The numbers k and m correspond to those introduced in Ref. 6. The linking number k is analogous and closely connected to the usual Hopf invariant of mappings $S^3 \to S^2$, given by $H=(1/4\kappa^2)\int \mathbf{v}_s \operatorname{curl} \mathbf{v}_s dV$ (Refs. 1 and 13). It turns out that the expression for k in terms of the superfluid velocity suggested in Ref. 6 is valid in this case. For arbitrary N it takes the form

$$\kappa=\frac{1}{4\kappa^2}\int \mathbf{v}_s \text{curl}\mathbf{v}_s dV-\frac{N}{2\kappa}\oint_{C^*}\mathbf{v}_s d\mathbf{r}, \tag{5}$$

and contains an additional integral over an arbitrary vertical generatrix C^* of the lateral surface (Fig. 2a) of the cell.

Intermediate rotational velocities. The vector $\mathbf{l}$ is no longer necessarily horizontal or constant at the boundary of any cross section. The only restriction is that it is not vertical ($\mathbf{l} \neq \pm \hat{z}$). The boundary conditions become periodic with vortex lattice periods. Nevertheless, constant boundary conditions will also be considered as a subcase which might be relevant in some other physical situation. For constant boundary conditions this case is equivalent to the previous case from the topological point of view; i.e., mappings are described by the indices N, m and k. Under periodic conditions there are two more invariants, m_1 and m_2, which are the winding numbers of the horizontal projection of the vector $\mathbf{l}$ along two nonparallel sides of the boundary ∂D^2 of a unit cell of the lattice (Fig. 2b). In order to give analytical expressions for k we need the notion of the area enclosed by a contour C on the $\mathbf{l}$-sphere.

This notion is apparently ill-defined. Still one can take any unit vector $\mathbf{n}$ such that C does not contain $\mathbf{n}$ and define the area Area $|_C^{\mathbf{n}}$ with respect to this vector to be $-\int_C(1+(\mathbf{ln}))d\phi/4\pi$, where ϕ is the longitude (the angle of rotations around $\mathbf{n}$). It gives the area swept by the arc of the great circle connecting $-\mathbf{n}$ and $\mathbf{l}$ which does not contain $\mathbf{n}$ (with the orientation taken into account). One can show that this quantity does not depend on $\mathbf{n}$ as it moves over the sphere, unless it crosses C. At this moment, the integral changes by an integer. It follows that only the fractional part of the area enclosed by C (in units of the whole area of the sphere) has an invariant meaning (we designate this fractional part $\mathrm{Area}_{|C}$). Let C^* be a contour in the real space, and $\mathbf{l}(C^*)$ its image on S^2. We also denote $\mathrm{Area}|_{\mathbf{l}(C^*)}$ by $\mathrm{Area}_{|C^*}$.

After these preliminary remarks we can write a more general formula for k:

$$k=\frac{1}{2\kappa}\oint_{C_\uparrow}\mathbf{v}_s d\mathbf{r}-\frac{N}{2\kappa}\oint_{C^*}\mathbf{v}_s d\mathbf{r}+N\mathrm{Area}|_{C^*}^{\downarrow}, \tag{6}$$

where $\downarrow$ denotes the south pole. An analogous formula, with $\uparrow$ and $\downarrow$ interchanged, is also valid. For $\mathbf{l}$ lying in the xy plane at the boundary we have $\mathrm{Area}|_{C^*}^{\downarrow} = -\mathrm{Area}|_{C^*}^{\uparrow} = m/2$.

We also give the following relation for the circulation of the superfluid velocity along $C_\mathbf{l}$ averaged over the whole sphere of vector $\mathbf{l}$:

$$\left\langle \oint_{C_\mathbf{l}}\mathbf{v}_s d\mathbf{r}\right\rangle=\frac{1}{2\kappa}\int \mathbf{v}_s \ \mathrm{curl}\ \mathbf{v}_s dV. \tag{7}$$

For the case of the vector $\mathbf{l}$, which is horizontal at the boundary, the integral over $C_\mathbf{l}$ on the left-hand side is constant on the northern and southern hemispheres. This permits us to link (6) and (5) with the help of Eq. (7).

Here is another equivalent formula for k in terms of $\int \mathbf{v}_s \mathrm{curl}\mathbf{v}_s dV$ (under the assumption $\mathbf{l}\neq\pm\hat{z}$ at the boundary):

$$k=\frac{1}{4\kappa^2}\int \mathbf{v}_s\mathrm{curl}\mathbf{v}_s dV-\frac{N}{\kappa}\oint_{C^*}\mathbf{v}_s d\mathbf{r}+\frac{N}{2\kappa}\left\langle \oint_{C^{**}}\mathbf{v}_s d\mathbf{r}\right\rangle+N(\mathrm{Area}|_{C^*}^{\downarrow}+\mathrm{Area}|_{C^*}^{\uparrow}). \tag{8}$$

Here C^* is a fixed vertical generatrix of the lateral boundary, C^{**} is an arbitrary vertical generatrix of the boundary, and the angle brackets denote the averaging over all such generatrices. To derive (8), we used the periodicity of $\mathbf{v}_s-\mathbf{\Omega}\times\mathbf{r}$ (Ref. 11).

Large rotational velocities. Let us assume that the angular velocity is large, and that $\mathbf{l}$ may assume arbitrary values at the boundary. The previous classification fails since the winding numbers m, m_1, and m_2 of the vector $\mathbf{l}$ cannot be defined. Depending on whether we assume the boundary conditions to be constant or periodic, we should now classify the mappings $S^2\times S^1\to S^2$ or $S^1\times S^1\times S^1\to S^2$ (Fig. 2). It can be shown (first done in Ref. 8) that a mapping $S^2\times S^1\to S^2$ is characterized by a number N (the number of quanta or the degree of the mapping of the horizontal cross section to $\mathbf{l}$-sphere) and, for a given N, by an element of Z_{2N} (i.e., integer modulo $2N$). For the mappings in the classes discussed above this element is given by $I=k+mN$ (mod $2N$). This means that configurations with the same N and $k+mN$ (mod $2N$) are topologically equivalent if we allow $\mathbf{l}$ to assume arbitrary values at the boundary. In particular, for $N=1$ the invariant is $k+m$ (mod 2); i.e., there are two types of configurations of the intersection of the 4π vortex with a soliton. For example, the configurations $m=k=1$ and $m=1$, $k=0$ mentioned in Ref. 6 are topologically different even in the limit of large angular velocities (this is also true for periodic boundary conditions; see the discussion below). For $N=0$

the invariant is an integer k: a localized defect in a uniform field of the vector $\mathbf{l}$ ($m=0$) or inside a twisted soliton ($m=1$) is characterized by the linking number of two loops $C_{\mathbf{l}_1}$ and $C_{\mathbf{l}_2}$.

Fix any $\mathbf{l}_0$ such that the field of the vector $\mathbf{l}$ does not take the value $\mathbf{l}_0$ at C^* (i.e., at the boundary). Then the invariant I is given by

$$I=\frac{1}{2\kappa}\oint_{C_{\mathbf{l}_0}}\mathbf{v}_s d\mathbf{r}-\frac{N}{2\kappa}\oint_{C^*}\mathbf{v}_s d\mathbf{r}+N\,\mathrm{Area}\Big|_{C^*}^{\mathbf{l}_0}\quad(\mathrm{mod}\ 2N),\tag{9}$$

or, in terms of $\int\mathbf{v}_s\,\mathrm{curl}\,\mathbf{v}_s dV$:

$$I=\frac{1}{4\kappa^2}\int\mathbf{v}_s\,\mathrm{curl}\,\mathbf{v}_s dV-\frac{N}{2\kappa}\oint_{C^*}\mathbf{v}_s d\mathbf{r}+2N\,\mathrm{Area}|_{C^*}\quad(\mathrm{mod}\ 2\mathrm{N}).\tag{10}$$

Mappings $S^1\times S^1\times S^1\to S^2$ are described, in general, by three integers N_1, N_2, and N_3, which show how many times three different faces of the cell cover the l-sphere, and by an element of $Z_{2\text{-GCD}(N_1,N_2,N_3)}$, where GCD denotes the greatest common divisor of N's. It follows, however, from periodicity of $\mathbf{v}_s-\mathbf{\Omega}\times\mathbf{r}$ (Ref. 11) that for the configuration in question only one of the three indices $\mathrm{N_i}$, which corresponds to the horizontal face, is nonzero. We denote it by $N=\mathrm{GCD}(N,0,0)$. The formula for the invariant is as follows [cf. Eq. (8)]:

$$I=\frac{1}{4\kappa^2}\int\mathbf{v}_s\,\mathrm{curl}\,\mathbf{v}_s dV-\frac{N}{\kappa}\oint_{C^*}\mathbf{v}_s d\mathbf{r}+\frac{N}{2\kappa}\left\langle\oint_{C^{**}}\mathbf{v}_s d\mathbf{r}\right\rangle$$
$$+2N\,\mathrm{Area}|_{C^*}\quad(\mathrm{mod}\ 2N).\tag{11}$$

In summary, we have discussed the topology of the intersection of a lattice of 4π-vortices and a soliton in ^{3}He-A and derived the analytical expressions for the proper invariants in terms of the distributions of the vector $\mathbf{l}$ and the superfluid velocity. We have proved using these invariants that the two competing configurations considered in Ref. 6 remain topologically different under any rotational velocity. The results also apply to other media with a broken symmetry described by a 3D vector, like the Heisenberg ferromagnets which are constrained by periodic or mixed periodic–constant boundary conditions.

We are indebted to G. E. Volovik for many discussions and interest in this work. Yu.G.M. acknowledges the International Science Foundation for the financial support, Grant MG1000. T.Sh.M. was partially supported by the Russian Foundation for Fundamental Research, Grants 93-02-02687 and 94-02-03121. T.Sh.M. also thanks the staff of Isaac Newton Institute for its kind hospitality.

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