On nonlinear MHD-stability of toroidal magnetized plasmas

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The variational approach to analyze the nonlinear magnetohydrodynamic (MHD) stability of ideal plasmas in magnetic fields of toroidal topology is proposed. The potential energy functional used as the Lyapunov functional is expressed in terms of a complete set of independent Lagrangian invariants, which allows one to take strictly into account all the restrictions inherent in the varied functions due to MHD dynamic equations. It is shown that no physical MHD perturbations can grow if the linear MHD stability is provided. © *1995 American Institute of Physics.*

FORMULATION OF THE PROBLEM

It is well known that the plasma stability problem is not exhausted by the conventional linear theory which cannot describe the perturbations that grow slower than exponentially.[1] To analyze the nonlinear plasma stability in the frame of ideal magnetohydrodynamics (MHD), it is very attractive to use the Lyapunov approach, choosing the plasma potential energy

$$W=\int d^3r\left(\frac{p}{\gamma-1}+\frac{B^2}{2}\right) \tag{1}$$

as the Lyapunov functional to be varied, because its time derivative cannot be positive due to the total energy conservation resulting from ideal MHD equations (see, e.g., Ref. 2). However, to avoid a possible narrowing of the class of analyzed equilibria and to obtain the harder stability criterion than it is necessary, certain restrictions on the varied functions must be taken into account. Such restrictions appear from the initial set of dynamic equations

$$(\partial_t+\mathbf{V}\nabla)\rho+\rho\,\mathrm{div}\mathbf{V}=0, \tag{2}$$

$$(\partial_t+\mathbf{V}\nabla)\frac{p}{\rho^\gamma}=0, \tag{3}$$

$$\partial_t\mathbf{B}=\mathrm{curl}[\mathbf{V}\times\mathbf{B}]. \tag{4}$$

The values p and ρ denote the pressure and density of the plasma which moves at a velocity $\mathbf{V}$ in the magnetic field $\mathbf{B}$, and γ is an adiabatic exponent. The quantities like ρ, which satisfy the continuity equation (2), are called Eulerian invariants, while the quantities like p/ρ^γ, which move together with the plasma, are called Lagrangian invariants.

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The above-mentioned restrictions can be added to the functional (1) as a set of Eulerian invariants. Since Eqs. (2)–(4) (with the motion equation) are known to have the infinite set of invariants, the earlier attempts to modify the functional (1) by adding an incomplete set of invariants led to the narrowing of the class of equilibria (see, e.g., Refs. 3 and 4). The adequate procedure which allows us to take strictly into account all the restrictions inherent in the varied functions under the integral (1) due to Eqs. (2)–(4) is described below.

Let us discuss in more detail the difference between our approach and the linear theory. Linear MHD stability is analyzed by a similar energy principle,[5] which states that for a linear instability it is necessary and sufficient to find a small plasma displacement ξ, provided that the quadratic form $\delta^2 W[\xi,\xi]$ is negative when the second variation of W is calculated using Eqs. (2)–(4) which are linearized over ξ. The point to apply to a nonlinear theory is that even here there are no linearly unstable perturbations. Nevertheless, there always are some "neutral" perturbations ξ_N which do not change the quadratic form given above:

$$\delta^2 W[\xi_N,\xi_N]=0.$$

Linear theory cannot say anything definite about the growth rate of such displacements. It can be determined only from the nonlinear analysis of the behavior of the potential energy in an equilibrium vicinity which is extended along those neutral displacements. Such an analysis must be distinguished from investigations of the nonlinear stages of linear instabilities.

CONSTRUCTION OF THE LYAPUNOV FUNCTIONAL

First of all, some obvious properties of the invariants, which result immediately from Eqs. (2)–(4), can be formulated:

A ratio of two Eulerian invariants is a Lagrangian invariant.

An arbitrary function of the Lagrangian invariants is also a Lagrangian invariant.

If $\mathbf{B}$ satisfies Eq. (4), the quantity $\mathbf{B}\nabla\alpha$ is a Eulerian invariant for any Lagrangian invariant α.

We introduce a system of independent coordinates $\{\mu,\nu,\lambda\}$, whose Jacobian $J=\nabla\mu[\nabla\nu\times\nabla\lambda]\neq 0$ everywhere in the plasma at a given moment of time. It is always possible "to glue" those coordinates to the plasma in such a way that they will obey Eq. (3), which means that they are Lagrangian invariants. The Jacobian can be easily proved to be a Eulerian invariant.[6] Therefore, it cannot be equal to zero at any time. Using Eqs. (2) and (3), we can construct the Eulerian invariant $p^{1/\gamma}$. As follows from the above-mentioned invariant properties, it can be expressed in terms of the Jacobian

$$p=J^{\gamma}\Pi(\mu,\nu,\lambda), \tag{5}$$

where the function Π specifies the pressure equilibrium distribution.

An arbitrary magnetic field satisfying (4) can be expressed in terms of the Lagrangian coordinates as follows:

$$\mathbf{B}=[\nabla\mu\times\nabla\nu]\mathscr{L}+[\nabla\nu\times\nabla\lambda]\mathscr{M}+[\nabla\lambda\times\nabla\mu]\mathscr{N},$$

where the functions $\mathscr{L}$, $\mathscr{M}$, and $\mathscr{N}$ of the Lagrangian coordinates must obey the single condition $\partial_\lambda \mathscr{L} + \partial_\mu \mathscr{M} + \partial_\nu \mathscr{N} = 0$ in order to provide $\mathrm{div}\mathbf{B}=0$. The closed magnetic field line system can be described by this formula by setting the two functions (e.g., $\mathscr{M}$ and $\mathscr{N}$) equal to zero identically. Considering the toroidal plasma only, we choose the coordinate μ as a magnetic surface mark:

$$\mathbf{B}\nabla\mu = 0, \tag{6}$$

which simply needs to set $\mathscr{M}=0$ in the general formula given above. Being satisfied at the initial moment, Eq. (6) remains valid at any time. Moreover, from our magnetic field representation it follows immediately that the topology of the magnetic field cannot be damaged during the time evolution (4). It should be mentioned that for any toroidal nested surfaces the coordinates $\{\mu,\nu,\lambda\}$ can be redefined to get the known Kruskal's flux representation for magnetic field:[7]

$$\mathbf{B} = [\nabla\tilde{\mu} \times (q(\tilde{\mu})\nabla\tilde{\nu} - \nabla\tilde{\lambda})], \tag{7}$$

where $\{\tilde{\nu},\tilde{\lambda}\}$ are, respectively, the analogs of the poloidal and toroidal angles in the tokamak, and $q(\tilde{\mu})$ is the ratio of the toroidal and poloidal fluxes (the "safety factor"). That representation is also valid at any time (not only at equilibrium), and is often more suitable for use due to its simpler form (the tildes are omitted below).

Substituting the explicit representations (5), (7) into the potential energy (1), we find that our functional (1) is expressed in terms of the Lagrangian invariants μ, ν, and λ without additional dependence on time. Now we may vary our functional over the coordinates μ,ν,λ independently, and there are no additional restrictions which must be taken into account.

FIRST AND SECOND VARIATIONS

We introduce the vector ξ by the relations

$$\delta\mu = -\xi\nabla\mu, \quad \delta\nu = -\xi\nabla\nu, \quad \delta\lambda = -\xi\nabla\lambda, \tag{8}$$

where δ denotes the corresponding variation. Having found the variations of the Jacobian $\delta J = -\mathrm{div} J\xi$, we can easily determine the variations of pressure and magnetic field:

$$\delta p = -\xi\nabla p - \gamma p\,\mathrm{div}\xi, \quad \delta\mathbf{B} = \mathrm{curl}[\xi\times\mathbf{B}], \tag{9}$$

which coincides precisely with the linear variations if the formally introduced vector ξ (8) plays the role of a linear plasma displacement. Equations (9) allow us to derive the first and second variations in the conventional form

$$\delta W \approx \int d^3r\,\xi(\nabla p + [\mathbf{B}\times\mathrm{curl}\mathbf{B}]),$$

$$\delta^2 W \approx \int d^3r((\delta\mathbf{B})^2 + \xi[\delta\mathbf{B}\times\mathrm{curl}\mathbf{B}] - \delta p\,\mathrm{div}\xi), \tag{10}$$

where δp and $\delta\mathbf{B}$ are given by Eqs. (9). Here and below only the contributions of integrals over the plasma volume to potential energy variations are considered for a brevity. The contribution of surface integrals results in the conventional conditions of plasma boundary equilibrium and transversality. Because of the arbitrariness of ξ, the condition $\delta W=0$ results in the general plasma equilibrium equation:

$$\nabla p+[\mathbf{B}\times \mathrm{curl}\mathbf{B}]=0. \tag{11}$$

It should be mentioned that in Refs. 3 and 4 instead of (11), only a narrow class of equilibria was obtained because of the incomplete accounting of the consequences of Eqs. (2)–(4).

The second variation $\delta^2 W$ in (10) coincides with the conventional energy principle[5] which is derived from the linear theory and which is obvious due to self-conjugation of the quadratic functional. However, the principal difference between the linearization procedure and our approach leads to a different treatment of the results obtained.

NEUTRAL DISPLACEMENTS

In contrast with the linear theory, our approach operates with the nonlinear energy functional (1) obtained from the precise nonlinear equation of motion. Furthermore, the proposed expressions (5) and (7) for the plasma pressure and magnetic field, given in terms of the independent Lagrangian invariants, represent all the relations between the variations of p and $\mathbf{B}$ due to the nonlinear dynamic equations (2)–(4). This means that the higher-order variations of the functional (1) can be also taken into account. Indeed, if $\delta^2 W \geqslant 0$ for any ξ, it does not necessarily provide a stability, because there always are some nontrivial neutral displacements ξ_N making $\delta^2 W|_{\xi_N} = 0$. These neutral displacements satisfy the Euler equation

$$\nabla\delta p+[\delta\mathbf{B}\times \mathrm{curl}\mathbf{B}]+[\mathbf{B}\times \mathrm{curl}\,\delta\mathbf{B}]=0, \tag{12}$$

which was used without fixing any norm of the displacments considered. The displacements ξ_N satisfying (12) may be divided into three classes:

1. $\xi_N\cdot\mathbf{n}\neq 0;\quad \xi_N\cdot\mathbf{n}|_S\neq 0\quad (\mathbf{n}=\nabla\mu/|\nabla\mu|)$.

Here S denotes the boundary magnetic surface. Such displacements correspond to a global equilibrium deformation (a case of near equilibria), and can be suppressed by external feedbacks, or a similar effect.

2. $\xi_N\cdot\mathbf{n}\neq 0;\quad \xi_N\cdot\mathbf{n}\Big|_S=0$.

Such displacements do not change certain equilibrium and correspond to the marginal stability situation, when the confined plasma is at the instability threshold. This situation can be broken down by slight changes of the equilibrium parameters, and is therefore of no interest to us.

3. $\xi_N\cdot\mathbf{n}=0$ everywhere.

Such displacements can take place at an arbitrary equilibrium, and therefore they must be analyzed first.

Looking for an explicit form of those displacements ξ_N and multiplying (12) by $\mathbf{B}$, we find $\delta\mathbf{B}\nabla p=-\mathbf{B}\nabla\delta p \Rightarrow \mathbf{B}\nabla \mathrm{div}\xi_N=0$. Since $\xi_N\ \mathbf{n}=0$, we have $\mathrm{div}\xi_N=0\Rightarrow\delta p|_{\xi_N} = 0$ and $\delta\mathbf{B}|_{\xi_N} = 0$, which results in the following expression for ξ_N:

$$\xi_N = a(p)\mathrm{curl}\mathbf{B} + b(p)\mathbf{B}, \tag{13}$$

where a and b are arbitrary functions [although for rational magnetic surfaces the form (13) must be modified; however, this fact cannot change the final conclusions given below].

Following our logic, we will investigate whether those neutral displacements can contribute to the higher-order variations of W.

THIRD AND FOURTH VARIATIONS

The reason to calculate the higher-order variations is that the neutral displacements may have a larger amplitude than the other displacements. Therefore, if their contribution to the $\delta^3 W$ is not zero, it will be able to compete with $\delta^2 W$ calculated over the significant displacements $\xi_* = \xi - \xi_N$ (different from the neutral ones). The volume part of

$$\delta^3 W \approx \int d^3r\{\delta^2\xi(\nabla p + [\mathbf{B}\times \mathrm{curl}\mathbf{B}]) + 2\delta\xi\delta(\nabla p + [\mathbf{B}\times \mathrm{curl}\mathbf{B}])$$
$$+ \xi(\nabla\delta^2 p + [\delta^2\mathbf{B}\times \mathrm{curl}\mathbf{B}] + 2[\delta\mathbf{B}\times \mathrm{curl}\,\delta\mathbf{B}] + [\mathbf{B}\times \mathrm{curl}\,\delta^2\mathbf{B}])\} \tag{14}$$

must be calculated using

$$\delta\xi\cdot\nabla\alpha = \xi\nabla(\xi\nabla\alpha), \tag{15}$$

where $\alpha = \{\mu,\nu,\lambda\}$ denotes any independent variable. Using the simple relations following from (15), we find

$$\delta_N\xi_N\cdot\nabla\mu = 0,$$
$$[\delta_N\xi_N\mathbf{B}] = 0, \tag{16}$$
$$\mathrm{curl}[\mathrm{curl}\mathbf{B}\times\xi_N] = 0$$

for $\delta_N\xi_N$: $\delta_N\xi_N\cdot\nabla\alpha = \xi_N\nabla(\xi_N\nabla\alpha)$. Substituting (16) into (14), we find that the largest part of $\delta^3 W \sim \|\xi_N\|^3$ is canceled identically, and that the term on the order of $\sim\|\xi_*\|\;\|\xi_N\|^2$ is

$$\delta^3 W[\xi_N,\xi_N,\xi_*] \approx \int 3\gamma p\,\mathrm{div}\xi_*\mathrm{div}(\delta_N\xi_N)d^3r.$$

This term can compete with $\delta^2 W \sim \|\xi_*\|^2$ if

$$\|\xi_N\| \sim \sqrt{\|\xi_*\|}. \tag{17}$$

The ordering (17) makes it necessary to calculate the fourth variation term $\sim\|\xi_N\|^4$:

$$\delta^4 W[\xi_N,\xi_N,\xi_N,\xi_N] \approx \int 3\gamma p\,\mathrm{div}^2(\delta_N\xi_N)d^3r.$$

Summing all the terms of the same order of magnitude, we find the change in the Lyapunov functional near the equilibrium:

$$\Delta W=\frac{1}{2}\,\delta^2W+\frac{1}{6}\,\delta^3W+\frac{1}{24}\,\delta^4W\approx\frac{1}{2}\int_{pl}$$

$$\times\left(\delta\mathbf{B}^2+\xi[\delta\mathbf{B}\times\mathrm{curl}\mathbf{B}]+\xi\nabla p\,\mathrm{div}\,\xi+\gamma p\,\mathrm{div}^2\left(\xi+\frac{1}{2}\,\delta_N\xi_N\right)\right)d^3r. \tag{18}$$

This formula differs from the expression for $\frac{1}{2}\delta^2W$ by a correction in the last compressible term. Due to (16) that correction can be canceled by redefining $\xi_\parallel$, without changing the other terms in (18).

FINITE NEUTRAL DISPLACEMENTS

The result of the previous section can be interpreted as a next-order correction of the neutral displacement. This means that the real neutral displacement has the form

$$\xi_N=\xi_N^0+\xi_N^1+\xi_N^2+\dots\,, \tag{19}$$

where ξ_N^0 is defined by (13), and $\xi_N^1=-\frac{1}{2}\delta_N\xi_N^0$. It is just a displacement that does not change the pressure and the magnetic field with counting of terms of order (17). The counting of higher-order variations results effectively in similar (18) corrections to the quadratic functional $\frac{1}{2}\delta^2W$, which corresponds to the next-order term in the expansion (19). Looking for finite neutral displacements instead of infinitesimal expansion (19), we find the following coordinate transform:

$$\alpha=\{\mu,\nu,\lambda\}\rightarrow\tilde{\alpha}=\{\tilde{\mu},\tilde{\nu},\tilde{\lambda}\}:$$

$$\mu=\tilde{\mu}; \tag{20}$$

$$F(\mu,\nu,\lambda)=F(\tilde{\mu},\tilde{\nu},\tilde{\lambda}),$$

where

$$F(\mu,\nu,\lambda)=F_1(\mu)\int_0^\nu\frac{d\zeta}{J_0[\mu,\zeta,\lambda+q(\zeta-\nu)]}+F_2(\mu,\lambda-q\nu),$$

and F_1 and F_2 are arbitrary functions (F_2 must only provide the partial derivatives of F to be the physical functions, 2π-periodic of angular-like variables ν,λ; such a function F_2 can always be found); J_0 is the Jacobian in the equilibrium state, which determines the function Π in (5) as a ratio $p_0(\mu)/J_0^\gamma(\mu,\nu,\lambda)$. It can be easily proved that the transform (20) provides the following relations between the initial ($J=\nabla_\mu[\nabla\nu\times\nabla\lambda]$) and the final ($\tilde{J}=\nabla\tilde{\mu}[\nabla\tilde{\nu}\times\nabla\tilde{\lambda}]$) Jacobians:

$$J=\tilde{J}\,\frac{J_0(\mu,\nu,\lambda)}{J_0(\tilde{\mu},\tilde{\nu},\tilde{\lambda})}$$

and $\tilde{\mathbf{B}}=\mathbf{B}$, $\tilde{p}=p$ due to the definitions (5), (7). Hence the transform (20) does describe the nontrivial, finite, neutral displacements, whose infinitesimal expansion (19) was found earlier.

DISCUSSION

The explicit representation of pressure (5) and magnetic field (7) by a set of independent Lagrangian invariants allowed us to vary the plasma potential energy by strictly taking into account all the relations derived from the MHD equations. That variational procedure resulted in the general equilibria and the stability criterion looking the same as in the linear stability theory. For *any* equilibrium there are infinitesimal neutral displacements (13), which do not change the potential energy to the second order (10). These infinitesimal neutral perturbations are extended to the finite displacements. From the mathematical point of view, the presence of such finite neutral displacements should be considered as an instability, because at least one of the coordinates $\{\nu,\lambda\}$ can increase infinitely. However, slow motion along such neutral directions does not change any macroscopic plasma parameters. There is no coupling of those neutral displacements of arbitrary amplitude with any other displacements. Therefore, from a physical point of view, they correspond to the coordinate relabeling and cannot be considered as a real instability. In other words, contrary to the conclusions of previous papers (e.g., Refs. 3 and 4), there is no nonlinear MHD instability for *any* static equilibria when the linear plasma stability is provided.

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