Solutions of the equations of the chiral-field model

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The equations of the model of 2D chiral fields with values in an arbitrary semisimple Lie group G are reduced to modified Nahm equations. A wide class of explicit solutions of these equations comes from solutions of the equations of a finite Toda chain.

1. In this letter we describe a new class of solutions of the equations of the model of 2D leading chiral fields. These models are important for studying nonperturbative effects in quantum field theory (Ref. 1, for example).

We write the equations of the model of leading chiral fields in the form proposed by Faddeev and Semenov-Tyan-Shanskiĭ.[2] For this purpose, we consider the vector fields A_μ and B_μ $(\mu,\nu,... = 1,2)$ in the space $R^{2,0}$ with values in the Lie algebra $\mathcal{G}$ of Lie group G. The equations for these fields are[2-4]

$$F_{\mu,\nu} = -[B_\mu, B_\nu], \qquad D_\mu B_\mu = 0, \qquad \epsilon_{\mu\nu} D_\mu B_\nu = 0, \tag{1}$$

where $F_{\mu,\nu} = [D_\mu, D_\nu]$, $D_\mu = \partial_\mu + [A_\mu,]$, and $\epsilon_{12} = -\epsilon_{21} = 1$. That these equations are equivalent to the standard equations of the chiral model was proved in Refs. 2 and 3.

Interestingly, solutions of Eqs. (1) simultaneously give us steady-state solutions of the nonlinear Schrödinger equation in the 3D space $R^{2,1}$ for a field Ψ in the associated representation of Lie algebra $\mathcal{G}$ which is interacting with gauge fields of Lie group G. This assertion was proved in Ref. 5. We also note that we can go over from the space $R^{2,0}$ to the space $R^{1,1}$ of signature $(+\ -)$. The entire discussion below for $R^{2,0}$ can be repeated with essentially no change for the case of $R^{1,1}$.

2. For the fields A_μ and B_μ we consider the ansatz

$$A_\mu = T_3(\varphi)\epsilon_{\mu\nu}\partial_\nu\varphi, \qquad B_\mu = T_1(\varphi)\epsilon_{\mu\nu}\partial_\nu\varphi - T_2(\varphi)\partial_\mu\varphi, \tag{2}$$

where the ϕ-dependent functions $T_a(\phi)$ $(a,b,... = 1,2,3)$ take on values in Lie algebra $\mathcal{G}$, and ϕ is an arbitrary function of the coordinates x_μ.

We substitute (2) into (1). One can easily verify that in this case Eqs. (1) reduce to the equations

$$T_a\Delta\varphi + (\dot{T}_a - f_a^{bc}T_bT_c)\partial_\mu\varphi\partial_\mu\varphi = 0, \tag{3}$$

where $f_1^{23} = f_2^{31} = -f_3^{12} = 1$, and $\Delta = \partial_\mu\partial_\mu$.

Equations (3) evidently hold if the following equations hold simultaneously:

$$\dot{T}_a = f_a^{bc} T_b T_c, \tag{4a}$$

$$\Delta \varphi = 0. \tag{4b}$$

Equations (4a) are known as modified Nahm equations. They can be found from Nahm's equations themselves[6] through the trivial substitutions $T_1 \to iT_1$, $T_2 \to iT_2$, $T_3 \to T_3$. Equation (4b) is the standard Laplace equation in $R^{2,0}$. We can take the real or imaginary part of an arbitrary analytic function as a solution of the latter equation.

3. Equations (4a) have a representation of the Lax type. To demonstrate this, we introduce the matrices $L(\lambda) = i(1+\lambda^2)T_1 - (1-\lambda^2)T_2 - 2i\lambda T_3$, $M(\lambda) = i\lambda T_1 + \lambda T_2 - iT_3$. Equations (4a) can then be rewritten as Lax equations with a spectral parameter λ (Refs. 7–9):

$$\dot{L}(\lambda) = [L(\lambda), \ M(\lambda)], \tag{5}$$

where $\dot{L} \equiv dL/d\phi$. It follows that the spectrum of matrix $L(\lambda)$ does not depend on ϕ. It also follows that the characteristic equation $\det[L(\lambda,\phi) - \eta I] = 0$, which determines the spectral curve, is an invariant of Eq. (5). The methods developed by Duborovin, Krichever, and Novikov (see, for example Ref. 10) can therefore be applied to Eq. (5), and we can write a general solution of Eq. (5) in terms theta functions.

A particular case of this class of solutions can be found by making use of Ward's observation[9] that it is possible to reduce Eqs. (4a) to equations of a finite Toda chain. The equations of a generalized Toda chain for an arbitrary simple Lie algebra were introduced by Bogoyavlensky.[11] Using the results of Ref. 11, we can easily write an ansatz for T_a in terms of a Cartan–Weyl basis of the Lie algebra $\mathcal{G} = \mathcal{G} \otimes \mathbb{C}$, which reduces (4a) to the equations of a Toda chain. As an example we write out an explicit expression for the ansatz for matrices T_a with values in Lie algebra $\mathcal{G} = su(n)$.

Following Ward,[9] we introduce the matrices h_j, e_j, e_{-j} ($j = 1,...,n$) with the components $(h_j)_{pq} = \delta_{j,p}\delta_{j,q}$, $(e_j)_{pq} = \delta_{j,p}\delta_{j+1,q}$ ($j = 1,...,n-1$), $(e_n)_{pq} = \delta_{1,p}\delta_{n,q}$, $(e_{-j})_{pq} = \delta_{j,p-1}\delta_{j,q}$ ($j = 1,...,n-1$), $(e_{-n})_{pq} = \delta_{n,p}\delta_{1,q}$. We set

$$T_1 = i\sum_{j=1}^{n} a_j(e_j + e_{-j}), \quad T_2 = \sum_{j=1}^{n} a_j(e_j - e_{-j}) - a_n(e_n - e_{-n}), \tag{6}$$

$$T_3 = i\sum_{j=1}^{n} b_j h_j, \qquad \sum_{j=1}^{n} b_j = 0,$$

where $a_j = a_j(\phi)$, $b_j = b_j(\phi)$ are real functions of ϕ. We introduce the matrices $L = i$-$T_1 - iT_3$, $M = T_2$. It is not difficult to verify that Eqs. (4a) can be rewritten as Lax equations for ansatz (6),

$$\dot{L} = [L, M], \tag{7}$$

in this case without a spectral parameter. These equations are the same as the equations of an ordinary finite periodic Toda chain (Refs. 11 and 12, for example). If we set $a_n = 0$ in (6), then Eqs. (7) become the equations of a finite aperiodic Toda chain, for which the explicit form of the general solutions is known.[12] We do not have room here to write out these solutions (see, for example, Refs. 12 and 13). We would like to point out that by formally taking the limit $n \to \infty$ in Eqs. (7) (this is the limit used in the quantum theory of chiral fields) we would obtain the standard equations of a infinite Toda chain, whose solutions are again known.

Ansatz (2) thus makes it possible to reduce the equations of a leading 2D chiral model to modified Nahm equations (4a), which can be integrated by methods of the inverse scattering problem, since it is a trivial matter to integrate Laplace equations (4b). The solutions of Eqs. (4a) and, in particular, of Eqs. (7) give us a new class of local solutions of the equations of the leading 2D chiral model and static solutions of the nonlinear Schrödinger equations in $R^{2,1}$. The question of the boundary conditions and that of the topological charge of these solutions require separate study.

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