Effective-range expansion and Coulomb renormalization in the $\alpha\alpha$, *dt*, and *d*³He systems

B. M. Karnakov, V. D. Mur, and S. G. Pozdnyakov
Moscow Engineering-Physics Institute, 115409, Moscow

V. S. Popov
Institute of Theoretical and Experimental Physics, 117259, Moscow

(Submitted 28 June 1991)
Pis'ma Zh. Eksp. Teor. Fiz. **54**, No. 3, 131–135 (10 August 1991)

An expression for the effective range derived here generalizes the Smorodinskiĭ formula to the case of potentials with a Coulomb repulsion and an arbitrary angular momentum *l*. With increasing charges of the particles, there is an exponential renormalization of the low-energy parameters. These parameters are extracted from experimental data for the *dt*, $d\,^3$He, and $\alpha\alpha$ systems.

1. The resonance nuclear reactions $dt \to n\alpha + 17.59$ MeV and $d\,^3\mathrm{He} \to p\alpha + 18.35$ MeV are important to problems in thermonuclear fusion, μ catalysis, astrophysics, etc. Their cross sections near the *s*-wave resonances $^5\mathrm{He}^*(3/2^+)$ and $^5\mathrm{Li}(3/2^+)$ have recently been measured with record-high accuracy.[1-4] It follows from an analysis of the experimental data that the Coulomb interaction causes a pronounced renormalization of not only the scattering length but also the effective range in these mirror systems (in contrast with the case of the *pp* and *pn* systems). The reason is that in this case the range of the nuclear forces, r_N, is comparable to the first Bohr radius.

2. Coulomb renormalization of the effective range. The expansion of the effective range for the charged particles is[5,6]

$$\frac{1}{a_B^{2l+1}}\Pi_{m=1}^{l}(\eta^{-2}+m^2)[2\pi D_C(\eta)\mathrm{ctg}\,\delta_l^{(cs)}+2h(\eta)]$$

$$\equiv K_l^{(cs)}(k^2) = -1/a_l^{(cs)} + \frac{1}{2}r_l^{(cs)}k^2 + \dots. \qquad (1)$$

Here $a_B = \hbar^2/Z_1Z_2e^2m$ is the first Bohr radius[1)] l is the angular momentum, $\eta = 1/ka_B$ is the Sommerfeld parameter, $k = (2E)^{1/2}$, $D_C(\eta) = [\exp(2\pi\eta) - 1]^{-1}$ is the penetrability of the Coulomb barrier,

$$h = \frac{1}{2}[\psi(i\eta) + \psi(-i\eta) - \ln\eta^2] \approx \sum_{j=1}^{\infty}\frac{1}{2j}|B_{2j}|\eta^{-2j}$$

($|\eta| \to \infty$, $-\pi/2 < \arg\eta < \pi/2$), B_{2j} are the Bernoulli numbers, and $a_l^{(cs)}$ and $r_l^{(cs)}$ are the nuclear-Coulomb scattering length and the effective range.

It can be shown that the value of $r_l^{(cs)}$ at the time at which a bound l level appears can be expressed in terms of the wave function χ_l with a zero energy:

$$r_l^{(cs)} = 2(2l-1)!!\int_0^\infty dr\left\{\left[\frac{c_l}{a_B^l}\xi_l(\rho)\right]^2 - \chi_l^2(r)\right\}$$

$$= a_B^{1-2l}/3(l!)^2 - 2(2l-1)!!\int_0^\infty \chi_l^2(r)dr, \qquad (2)$$

$c_l^2 = 2^l(4l+3)/3(2l+1)[l!(2l+1)!]$. At $r \gg r_N$, i.e., outside the range of the nuclear forces, we have $\chi_l(r) = r^{-l}\xi_l(\rho)$, where

$$\xi_l(\rho) = \frac{\rho^{2l+1}K_{2l+1}(\rho)}{2^{2l}(2l)!} \; (\rho \gg 1) \approx \frac{\pi^{1/2}}{(2l)!}\left(\frac{\rho}{2}\right)^{2l+1/2} e^{-\rho} + \ldots,$$

$\rho = (8r/a_B)^{1/2}$, and $K_\nu(\rho)$ is the modified Bessel function [the decay of $\chi_l(r)$ as $r \to \infty$ for all l, including $l=0$, stems from the Coulomb barrier]. "Turning off" the Coulomb interaction corresponds to $a_B \to \infty$ and $\rho \to 0$. Using $\xi_l(0) = 1$, we find that relation (2) becomes the Smorodinskiĭ formula[6,7] ($l=0$) and the formula of Ref. 8 ($l \geq 1$) for short-range potentials in this limit.

Let us take a closer look at the case of s scattering. We denote by R_C the minimum distance at which the strong interaction is still negligible in comparison with the Coulomb interaction. We then find from (2) a limitation which is useful for extracting r_{cs} from experimental data:

$$r_{cs} \leq a_B H(\rho_C), \qquad (3)$$

where $\rho_C = (8R_C/a_B)^{1/2}$ and

$$H(\rho) = \frac{1}{3} - \frac{1}{2}\int_\rho^\infty \xi_0^2(t)t\,dt \; (\rho \gg 1) \approx \frac{1}{3} - \frac{\pi}{8}e^{-2\rho}(\rho^2 + \frac{7}{4}\rho + \ldots) \qquad (3')$$

(Fig. 1). We thus see that at $r_N \gg a_B$ the effective range is exponentially close to its limiting value of $a_B/3$.

A corresponding assertion holds for higher-order coefficients of expansion (1). For example, for s-wave scattering one can show, by a semiclassical method, that in the limit $r_N \gg a_B$ we have

$$K_0^{(cs)}(k^2) = \sum_{j=0}^\infty \alpha_j k^{2j} \to 2h(\eta)$$

while for $j \geq 1$ the following estimate is valid:

$$\Delta\alpha_j = j^{-1}|B_{2j}| - \alpha_j = O(\delta), \quad \delta = 2\pi\exp[-(32r_N/a_B)^{1/2}] \ll 1. \qquad (4)$$

The reason why $\Delta\alpha_j$ is exponentially small is that the Coulomb barrier is cut off at $r \leq r_N$.

3. Low-energy parameters of $\alpha\alpha$ scattering. In the $\alpha\alpha$ system there is a narrow

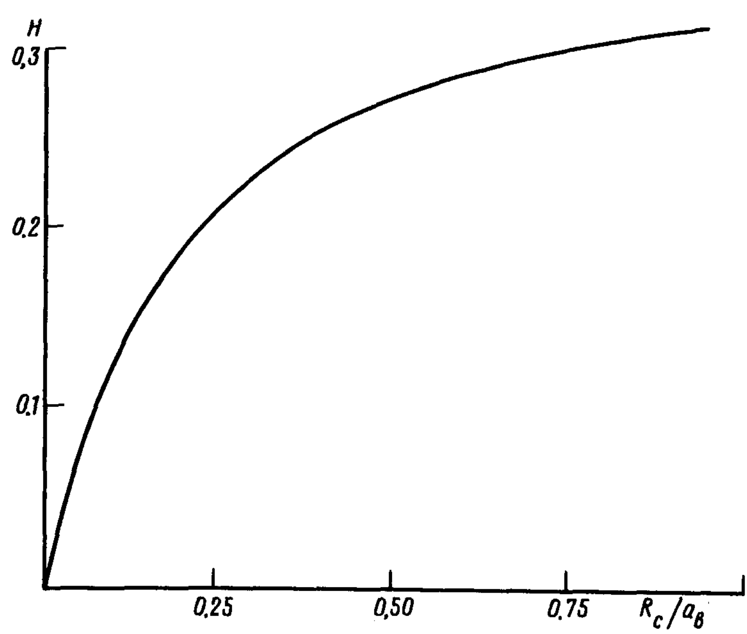

FIG. 1. Plot of the function H [see Eq. (3')] versus $R_C/a_B = \rho_C^2/8$.

Breit–Wigner resonance, ^{8}Be(0^+), with an energy[9] $E = E_r - i\Gamma/2$, where $E_r = 92.12 \pm 0.05$ keV and $\Gamma = 6.8 \pm 1.7$ eV. Here $r_N \approx a_B$ and $\delta = 2.8 \times 10^{-2}$. Consequently, we assume, in accordance with (4),

$$2\pi D_C \mathrm{ctg}\delta_{C0} = \alpha_0 - \Delta\alpha_1 k^2 - \Delta\alpha_2 k^4 - \Delta\alpha_3 k^6 + \dots . \quad (5)$$

Using the experimental data in Ref. 10 on the scattering phase shift for $E < 1000$ keV (or, correspondingly, $kr_N < 1$), using the value of E_r, and setting $\Delta\alpha_3 = 0$, we find the optimum set of parameters α_i, which corresponds to $\chi^2 = 0.33$ (Table I). In particular, we find $\Gamma = 3.6$ eV as the width of the "ground" state of ^{8}Be; this result agrees within two standard deviations with the experimental value. Extracting a more reliable value of Γ from scattering data will require an improvement in the experimental accuracy at $E < 400$ keV.

4. The mirror systems d^{3}H and d^{3}He. If a system contains open channels, the expansion coefficients in (1) are complex. For $l = 0$ we write $K_{Cs} = \alpha(k^2) - i\beta(k^2)$, where

$$\alpha(k^2) = \alpha_0 + \alpha_1 k^2 + \alpha_2 k^4 + \dots, \quad \beta(k^2) = \beta_0 + \beta_1 k^2 + \beta_2 k^4 + \dots .$$

It then follows from the unitarity condition that we have $\beta(k^2) \geqslant 0$.

The resonance s wave plays a dominant role in elastic dt and $d\,^3$He scattering and also in fusion reactions. Restricting the discussion to that wave, we find the following result for the astrophysical function:[11]

TABLE I.

System	dt	d^3He	$\alpha\alpha$
a_B	24,04	12,02	3,627
r_N	3,63	3,97	3,34
a_{cs}	$76+i\cdot 31$	$65+i\cdot 7,3$	1,19(3)
r_{cs}	$4,9-i\cdot 0,3$	$3,3-i\cdot 0,2$	1,02
R_C	5,6	6,3	2,0
α_0	0,270	0,184	3,05(-3)
β_0	0,110	2,08(-2)	0
α_1	0,102	0,138	0,141
β_1	6,98(-3)	8,57(-3)	0
α_2	-	5,0(-3)	1,16(-2)

Note. The values of a_B, r_N, etc., which have the dimensionality of a length, are in femtometers. Here the sum of the charge radii of the particles is used as r_N. The order of the number is given in parentheses: $(n)\equiv 10^n$.

$$s(E)=\beta(k^2)\{[\alpha(k^2)-2h(\eta)]^2+[\beta(k^2)+2\pi D_c(\eta)]^2\}^{-1}. \quad (6)$$

Using the experimental data of Refs. 1–4, we find sets of low-energy parameters for the dt and d^3He systems (Table I). The quality of the fit is illustrated by Fig. 2.

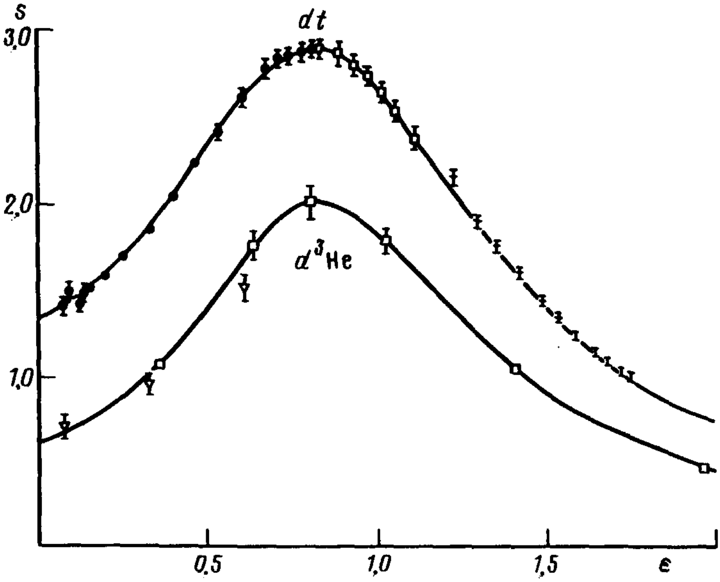

FIG. 2. The astrophysical function (6) versus $\varepsilon=E/E_C$. The solid lines are calculations from (6) with the parameter values from Table I. ●—Experimental data from Ref. 1; ○—Ref. 2; +—Ref. 12; ▽—Ref. 3; □—Ref. 4.

TABLE II. Energies (in keV) of the low-energy resonances.

System	dt	d^3He	$\alpha\alpha$
E_R	$47.2 - i \cdot 37.7$	$160 - i \cdot 118$	$92.1 - i \cdot 1.8(-3)$
ν_R	$0.233 - i \cdot 0.665$	$0.210 - i \cdot 0.693$	$2.86(-5) - i \cdot 2.937$
E_C	59.89	239.5	1588

A matched variation of all the parameters α_i, β_i makes possible a variation of these values over a fairly wide region with an inconsequential increase in χ^2. As a further test of the selection, we used limitation (3) on the range of the nuclear interaction. In the case of dt scattering, for example, we have the following result for an optimum choice (optimum in terms of a minimum of χ^2, $\chi^2 = 0.62$): $r_{cs} = 5.7$ fm. This result corresponds to $R_c > 8$ fm (Fig. 1). Such a value of R_c looks physically reasonable (in the R-matrix approach, it is customary to assume $R_c \approx 5$ fm for this system). The value of r_{cs} in Table I corresponds to $\chi^2 = 0.82$.

It can be seen from Table I that the Coulomb interaction in the mirror systems $d\,^3$H and $d\,^3$He causes a substantial renormalization of not only the scattering length (this point is well known[6]) but also the effective range r_{cs}. Correspondingly, the positions of the ^{5}He* and ^{5}Li* resonances are also shifted by an amount on the order of the Coulomb energy (Table II). The energies of these resonances,[2)] expressed in Coulomb units, $E_C = Z_1^2 Z_2^2 e^4 m/\hbar^2$, are fairly close together, as can be seen particularly clearly when we use the dimensionless variables $\nu = -i(2E/E_C)^{-1/2}$, which is a generalization of the principal quantum number (for virtual levels in a repulsive Coulomb field we would have $k = -i/n$ and $\nu = n = 1,2,...$).

[1)] Below we discuss the case of a Coulomb repulsion ($Z_1 Z_2 > 0$), and we use a system of units with $\hbar = m = a_B = 1$.

[2)] The values of E_R and ν_R in Table II correspond to the so-called leading pole. In addition to that pole, the scattering amplitude has a pole R' as well as a Coulomb series of poles,[11] which condense on the elastic limit ($k = 0$).

[1]N. Jarmie *et al.*, Phys. Rev. C **29**, 2031 (1984).

[2]R. E. Brown *et al.*, Phys. Rev. C **35**, 1999 (1987).

[3]W. Möller and R. Besenbacher, Nucl. Instrum. Meth. **168**, 111 (1980).

[4]A. Krauss *et al.*, Nucl. Phys. A **465**, 150 (1987).

[5]E. Lambert, Helv. Phys. Acta **42**, 667 (1969).

[6]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, Nauka, Moscow, 1989 (previous editions of this book have been published in English translation by Pergamon, Oxford).

[7]Ya. A. Smorodinskiĭ, Dokl. Akad. Nauk SSSR **60**, 217 (1948).

[8]V. D. Mur and V. S. Popov, Teor. Mat. Fiz. **27**, 204 (1976).

[9]J. Benn *et al.*, Nucl. Phys. A **106**, 296 (1968).

[10]N. P. Heydenburg and G. M. Temmer, Phys. Rev. **104**, 123 (1956).

[11]B. M. Karnakov *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **51**, 352 (1990) [JETP Lett. **51**, 399 (1990)].

[12]A. P. Kobzer *et al.*, Yad. Fiz. **3**, 1060 (1966) [Sov. J. Nucl. Phys. **3**, 774 (1966)].

Translated by D. Parsons