# "Wave-number conservation" and succession of symmetries during a Whitham averaging

V. R. Kudashev
*Bashkir State University, 450074, Ufa*

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A simple relationship has been observed between Whitham's equations, which have Riemannian invariants, and "wave-number conservation laws," which arise when an averaging is carried out by Whitham's method.

**1.** Whitham's method[1] has proved effective in describing several nonlinear processes (the best-known of which is a formulation of the Gurevich–Pitaevskiĭ problem[2]). This success, combined with the discovery of the generalized hodograph method,[3,4] has spurred a large number of studies in this field (see, e.g., Refs. 4–7 and the bibliographies there).

Our main purpose in this letter is to report a simple relationship which holds between Whitham's equations in "Riemannian form" and conservation laws for these equations which are of a "wave-number-conservation" type. One of the simplest consequences of this relationship is a physically transparent formulation of a principle of the succession of higher symmetries when single-phase periodic solutions of several nonlinear equations are averaged by Whitham's method.

**2.** We recall that Whitham's equations in Riemannian form generally fall in the category of semi-Hamiltonian systems of a hydrodynamic type,[4,5] i.e., diagonal systems of the type (a repeated index does *not* imply a summation)

$$r^i_t = \varphi^i(r) r^i_x, \quad r^i = r^i(t,x), \quad i = 1, \dots, N, \tag{1}$$

where $r^i_t = D_t r^i \equiv dr^i/dt$, $r^i_x = D_x r^i \equiv dr^i/dx$, and $\varphi^i \neq \varphi^k$. In addition, the $\varphi^i(r)$ satisfy the relations

$$\partial_j c^{ik} = \partial_k c^{ij}, \quad \partial_j \equiv \partial/\partial r^j, \quad (i \neq j \neq k \neq i), \tag{2}$$

where

$$c^{ik} = (\partial_k \varphi^i)/(\varphi^k - \varphi^i), \qquad (i \neq k). \tag{3}$$

We know[4,5] that conservation laws of a hydrodynamic type hold for Eqs. (1):

$$D_t \rho(r) = D_x \sigma(r), \tag{4}$$

where the density $\rho$ and the current $\sigma$ are functions which depend on $r^i$ but not on $r^i_x, r^i_{xx}, \dots$ .

The observation we wish to make is that if a conservation law (4) is known, i.e., if we know a $(\rho,\sigma)$ pair with $(\partial_i \rho) \neq 0$, then Eqs. (1) can be written in the form

  

$$(\partial_i \rho) r_t^i = (\partial_i \sigma) r_x^i, \quad i = 1, \ldots, N. \tag{5}$$

Rewriting (4), we have

$$D_t \rho = \sum_i (\partial_i \rho) \varphi^i r_x^i = \sum_i (\partial_i \sigma) r_x^i. \tag{6}$$

With $r_x^i \neq 0$ we find from (6)

$$\varphi^i = (\partial_i \sigma)/(\partial_i \rho), \qquad i = 1, \ldots, N, \tag{7}$$

from which we in turn find (5). The opposite is also true. If we know $(\rho, \varphi^i)$, then the current $\sigma$ can be reconstructed from the combined system of equations [combined under condition (2)]

$$\partial_i \sigma = \varphi^i (\partial_i \rho), \qquad i = 1, \ldots, N. \tag{8}$$

If $\partial_i \varphi^i = 0$, then we find simply $\sigma = \varphi^i \rho$ from (8) (Ref. 8).

In an analysis of (1) by the generalized hodograph method, the hydrodynamic symmetries,[4] i.e., the equations

$$r_\tau^i = \psi^i(r) r_x^i, \qquad r^i = r^i(\tau, t, x), \qquad i = 1, \ldots, N, \tag{9}$$

which commute with (1) $(\partial_\tau r_t^i = \partial_t r_\tau^i)$, are important, as are the hydrodynamic conservation laws for (9):

$$D_\tau \rho(r) = D_x \bar{\sigma}(r), \tag{10}$$

where $\rho$ is the same density as in (4). By analogy with (5), in terms of the pair $(\rho, \bar{\sigma})$ we have for (9)

$$(\partial_i \rho) r_\tau^i = (\partial_i \bar{\sigma}) r_x^i, \qquad i = 1, \ldots, N, \tag{11}$$

where $\bar{\sigma}$ satisfies the joint system of equations

$$\partial_i \partial_k \bar{\sigma} = \bar{c}^{ik} \partial_i \bar{\sigma} + \bar{c}^{ki} \partial_k \bar{\sigma}, \qquad (i \neq k), \tag{12}$$

where

$$\bar{c}^{ik} = c^{ki} (\partial_k \rho)/(\partial_i \rho), \qquad \partial_j \bar{c}^{ik} = \partial_k \bar{c}^{ij}, \qquad (i \neq k \neq j \neq i). \tag{13}$$

**3.** As we know, the Korteweg–de Vries equation

$$u_t + u u_x + u_{xxx} = 0 \tag{14}$$

and its higher symmetries (Ref. 9, for example)

$$u_{t_{2n+1}} + D_x R^n u = 0 \tag{15}$$

$(n = 1,2,\ldots;\ R = D_x^2 + 2u/3 - D_x^{-1} u_x/3;\ t_3 = t)$ have a single-phase periodic solution (a cnoidal wave) with a fixed period[1,2,5,6,10] $u = \varphi(\theta) = \varphi(\theta + 1)$:

$$\varphi(\theta) = 2adn^2(2K(m)\theta, m) + r_1 + r_2 - r_3. \quad (16)$$

Here $a = r_3 - r_1$, $m = (r_2 - r_1)/(r_3 - r_1)$ is the modulus of the Jacobi function $dn$, $K(m)$ is the complete elliptic integral of the first kind, $r_1 \leqslant r_2 \leqslant r_3$,

$$\theta_x = \kappa = \frac{(a/6)^{1/2}}{2K(m)}, \quad (17)$$

$$\theta_{t_{2n+1}} = \omega_{2n+1} = -\kappa U_{2n+1}, \quad (18)$$

where $\omega_3 \equiv \omega$, $U_3 \equiv U = (r_1 + r_2 + r_3)/3$ (recurrence relations for the other $U_{2n+1}$ are given in Ref. 10).

Assuming that $r^i$ are slowly varying functions of $(x, t_{2n+1})$, following Whitham's method,[1] and working in the standard way, we find from (17) and (18) some commuting conservation laws for a "wave number":

$$\partial\kappa/\partial t_{2n+1} = \partial\omega_{2n+1}/\partial x, \qquad (n = 1, 2, \ldots). \quad (19)$$

Since we know that the quantities $r^i$ are Riemannian invariants,[1,2,5] we find from (5), (11), and the wave conservation laws in (19) that, when the cnoidal waves of the Korteweg–de Vries equation are averaged by Whitham's method, the higher symmetries in (15) succeed one another in accordance with (cf. Refs. 5 and 10)

$$(\partial_i\kappa) r^i_{t_{2n+1}} = (\partial_i\omega_{2n+1}) r^i_x, \qquad (i = 1, 2, 3; \quad n = 1, 2, \ldots). \quad (20)$$

With $n = 1$ we find the ordinary Whitham equations[1,2,5,6] from (18), in the form

$$(\partial_i\kappa) r^i_t = (\partial_i\omega) r^i_x, \qquad (i = 1, 2, 3). \quad (21)$$

If we substitute $\omega_{2n+1}$ from (18) into (20), we find[10]

$$r^i_{t_{2n+1}} + (Q_i U_{2n+1}) r^i_x = 0, \qquad (i = 1, 2, 3; \quad n = 1, 2, \ldots), \quad (22)$$

where

$$Q_i = 1 + \frac{\kappa}{(\partial_i\kappa)} \frac{\partial}{\partial r_i}, \qquad (i = 1, 2, 3). \quad (23)$$

Clearly, the form of Whitham's equations (20)–(23) and their relationship with the wave-number conservation laws in (19) can be extended in a natural way to Whitham's equation (for which Riemannian invariants are known) which arise when an average is taken of the single-phase periodic solutions of other nonlinear equations. Examples of the use of Whitham's equations and of the hydrodynamic symmetries in the form in (22) in the case of the nonlinear Schrödinger equation and the sine-Gordon equation are given in Ref. 11.

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