Topological phases for a system of three mixed Dirac neutrinos in a medium of varying density

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Three-neutrino oscillations in a medium with arbitrary smooth distributions of the density and the composition are analyzed. Berry's adiabatic approach is used to study the conditions for the appearance of topological phases and their relationship with the *CP*-breaking phase.

The appearance of topological phases is a general property[1] of the Schrödinger evolution of dynamic systems with Hamiltonians which depend on the time through a set of adiabatic parameters $\{\lambda_\alpha(t)\}$. These phases appear in the amplitudes for quantum transitions, in a sum with dynamic phases. Neutrino oscillations in a medium with a varying density[2] (the density depends on the coordinate $x \simeq ct$) are typical examples of such an evolution. In this case the role of the parameters λ_α is played by the refractive indices for neutrinos in the medium, n_α, or combinations thereof. It is legitimate to ask about the particular conditions under which topological phases will arise in a neutrino system and about the effect of these phases on the oscillation probabilities. If absorption is ignored, and if a possible contribution from off-diagonal neutral currents is ignored,[3] topological phases can arise only in a system with $N \geqslant 3$ mixed neutrinos, since the vacuum mixing matrix V and therefore the Hamiltonian of a 2ν system are real.

In this letter we consider the very simple case of three mixed Dirac neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ as they propagate through a medium with "smooth" and otherwise arbitrary distributions of the density and the composition. The equation for the evolution operator[1)] $S(t) = ||S_{\alpha\beta}(t)||$ of a system of this sort[3] can be written

$$i\dot{S}(t) = H[\vec{q}(t)]S(t), \qquad (1)$$

with the initial condition $S(0) = I$. The Hamiltonian in (1),

$$H[\vec{q}] = \begin{pmatrix} \mathcal{W}_e - q_e & \mathcal{X}_\tau & \mathcal{X}_\mu^* \\ \mathcal{X}_\tau^* & \mathcal{W}_\mu - q_\mu & \mathcal{X}_e \\ \mathcal{X}_\mu & \mathcal{X}_e^* & \mathcal{W}_\tau - q_\tau \end{pmatrix},$$

depends on the time only through the components of the vector $\vec{q} = (q_e, q_\mu, q_\tau)$, which are related to the refractive indices n_α:

$$q_\alpha(t) = p_\nu[n_\alpha(t) - \langle n(t) \rangle], \quad \langle n(t) \rangle = \frac{1}{3}\sum_\alpha n_\alpha(t).$$

Here p_ν is the neutrino momentum $[p_\nu \gg \max(m_i), c = 1]$. We will be assuming that continuous derivatives $\dot{q}_\alpha$ exist for all t, although this limitation is not necessary. The

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constants $\mathscr{W}_\alpha$ and $\mathscr{H}_\alpha$ are determined by the matrix $V=||V_{\alpha i}||$, which generally depends on the three mixing angles θ_i and on the CP-breaking parameter δ (the "Dirac phase"), and by the masses m_i of the states $|\nu_i\rangle = \Sigma_\alpha V^*_{\alpha i}|\nu_\alpha\rangle$:

$$\mathcal{W}_\alpha = \sum_i |V_{\alpha i}|^2\Delta_i, \quad \mathcal{H}_\alpha = \eta_\alpha^{\beta\gamma}\sum_i V_{\beta i}V^*_{\gamma i}\Delta_i,$$
$$\Delta_i = \frac{m_i^2 - \langle m^2\rangle}{2p_\nu}, \qquad \langle m^2\rangle = \frac{1}{3}\sum_i m_i^2.$$

Here $\eta_\alpha^{\beta\gamma}=1$, if $(\alpha\beta\gamma)$ is a cyclic permutation of the indices $(e\mu\tau)$, and $\eta_\alpha^{\beta\gamma}=0$ otherwise.[2)] The Hamiltonian H is Hermitian, and the matrix S is unitary, since we are assuming $\mathrm{Im} n_\alpha(t)=0$.

We denote by $\varepsilon_j|\vec{q}|$ eigenvalues and by $|\mathscr{U}_j|\vec{q}|\rangle$ the corresponding orthonormal eigenvectors of the Hamiltonian $H|\vec{q}|$. It can be shown that the levels $\mathscr{E}_j$ do not intersect in an arbitrary evolution of a 3ν system, provided that the dynamic variable

$$J = \mathrm{Im}\prod_\alpha\sum_i \eta_\alpha^{\beta\gamma}U_{\beta i}U^*_{\gamma i}\mathcal{E}_i = \mathrm{Im}\prod_\alpha \mathcal{H}_\alpha$$

is nonzero. Here $U=||U_{\alpha i}||=(|\mathscr{U}_1\rangle,|\mathscr{U}_2\rangle,|\mathscr{U}_3\rangle)$ is the neutrino mixing matrix in the medium. This means that in the case $J\neq 0$ only Abelian topological phases can appear.[1] On the other hand, we have $J=0$ if there is no mixing and also if the case reduces to one of two-neutrino oscillations (if $\Sigma_\alpha|\mathscr{H}_\alpha|^2\neq 0$). At $J=0$, topological phases thus do not arise.

According to Berry,[1] the Abelian topological phases γ_j are given by the integrals

$$\gamma_j[\vec{q}(t)] = \int_C \mathcal{A}_j[\vec{q}]d\vec{q},$$

where $\mathscr{A}_j[\vec{q}]$ is a gauge field with the components (Berry constraints) $\mathscr{A}_j^\alpha = i\langle\mathscr{U}_j|\partial/\partial q_\alpha|\mathscr{U}_j\rangle$, and the integration is carried out between the points $q(0)$ and $q(t)$, along a contour C which lies in the $Q=\{\vec{q}|\Sigma_\alpha q_\alpha=0\}$ plane. Imposing the condition $\Sigma_\alpha \arg(U_{\alpha j}) = \Sigma_\alpha \arg(V_{\alpha j})$ on the common phase $|\mathscr{U}_j\rangle$, and finding explicit expressions[3)] for the quantities $U_{\alpha j}$, we find

$$\mathcal{A}_j^\alpha = (J/3\xi_j)\eta_\alpha^{\beta\gamma}(1/\xi_j^\beta - 1/\xi_j^\gamma),$$

$$\xi_j^\alpha = \eta_\alpha^{\beta\gamma}(\mathcal{E}_j - \mathcal{W}_\beta + q_\beta)(\mathcal{E}_j - \mathcal{W}_\gamma + q_\gamma) - |\mathcal{H}_\alpha|^2, \quad \xi_j = \sum_\alpha \xi_j^\alpha.$$

It can be shown that in the case $J\neq 0$ the functions ξ_j^α have no zeros, so the field $\mathscr{A}_j[\vec{q}]$ is regular in the Q plane. We of course have $\mathscr{A}_j^\alpha\sim J$ only in the selected gauge (which we call the "J gauge"). When we go over to a basis $|\widetilde{\mathscr{U}}_j\rangle = \exp\{i\chi_j[\vec{q}]\}|\mathscr{U}_j\rangle$, which is equivalent in the unitarity sense, the constraints $\mathscr{A}_j^\alpha$ and phases γ_j transform in the following way:

$$\mathcal{A}_j^\alpha \to \mathcal{A}_j^\alpha - \partial\chi_j/\partial q_\alpha, \quad \gamma_j \to \gamma_j - \chi_j[\vec{q}(t)] + \chi_j[\vec{q}(0)].$$

An obvious gauge-invariant consequence of the nonsingular nature of the field $\mathscr{A}_j[\vec{q}]$ is that the Berry phases which arise upon a periodic variation of the parameters of the medium vanish:

$$\gamma_j^B \equiv \gamma_j[\vec{q}(T)] = \oint_C A_j d\vec{q}, \quad \vec{q}(T) \overset{def}{=} \vec{q}(0).$$

The meaning here is that the topological phases $\gamma_j[\vec{q}(t)]$ do not depend on the distributions of the density and the composition between the points $x(0)$ and $x(t)$ in the medium.

To determine the physical meaning of the J gauge, it is useful to write the explicit expression for J in the Kobayashi–Maskawa representation:[4]

$$J = \sin\delta \sin\theta_1 \prod_i \left[\eta_i^{jk} \frac{1}{4p_\nu}(m_j^2 - m_k^2)\sin 2\theta_i\right]. \tag{2}$$

We see from (2) that we have $\gamma_j = 0$ (in the J gauge) if any of the following conditions holds: (a) Any of the mixing angles θ_i is equal to 0 or $\pi/2$; (b) the Dirac phase δ is a multiple of π (there is no CP breaking); (c) the neutrino mass spectrum is degenerate.

Let us examine the solution of evolution equation (1) in the adiabatic approximation, which is important in many applications of the theory of neutrino oscillations in matter.[2] It is easy to verify that the adiabatic evolution operator of a 3ν system is

$$S^A(t) = U[\vec{q}(t)]D(-\vec{\Omega}(t))U^+[\vec{q}(0)], \tag{3}$$

where $D(\vec{\Omega})$ is a unitary diagonal matrix: $D(\vec{\Omega}) = ||\exp(i\Omega_j)\delta_{jk}||$, where $\Omega_j(t) = \phi_j(t) - \gamma_j[\vec{q}(t)]$ are the total phases, and $\phi_j(t)$ are the "dynamic" phases, given by

$$\phi_j(t) = \int_0^t \mathcal{E}_j[\vec{q}(\tau)]d\tau.$$

It follows from (3) that the dynamic and topological phases are completely equivalent in terms of determining the operator S^A. Since this operator is gauge-invariant, the topological effects cannot be eliminated by any redefinition of the basis $\{|\mathscr{U}_j\rangle\}$. It is fairly clear that a corresponding assertion holds in the general case of a nonadiabatic evolution. According to (3), the probability for an adiabatic transition $\nu_\alpha \to \nu_\beta$ over a time t is

$$P_{\alpha\beta}(t) = \sum_i |U_{\alpha i}^{\circ} U_{\beta i}^t|^2 + 2\mathrm{Re}\sum_i \eta_i^{jk} U_{\alpha j}^{\circ} U_{\beta k}^t (U_{\alpha k}^{\circ} U_{\beta j}^t)^* \exp(i\Omega_{jk}^t),$$

where we have used the shorthand

$$U_{\alpha i}^t = U_{\alpha i}[\vec{q}(t)], \quad \Omega_{jk}^t = \Omega_j(t) - \Omega_k(t).$$

A case of particular interest is that in which the properties of the medium vary periodically: $\vec{q}(t+T)=\vec{q}(t)$. Approximately this case is realized when (in particular) a neutrino flux intersects the Earth. This case is important for analyzing underground neutrino experiments. By virtue of the identity $\gamma_j^B=0$, topological effects do not influence the transition amplitudes over a time equal to an integer number (K) of periods T, and the evolution operator is given by the very simple expression

$$S^A(KT)=U[\vec{q}(0)]D(-K\vec{\phi}(T))U^+[\vec{q}(0)]. \tag{4}$$

From (4) we find expressions for the transition probabilities which generalize the corresponding results of Ref. 5 for a medium with a constant density:

$$P_{\alpha\alpha}(KT)=1-4\sum_i \eta_i^{jk}\left\{|U^\circ_{\alpha j}U^\circ_{\alpha k}|\sin\left(\frac{1}{2}K\phi^T_{jk}\right)\right\}^2,$$

$$P(KT)\equiv\eta_\alpha^{\beta\gamma}(P_{\beta\gamma}(KT)-P_{\gamma\beta}(KT))=4J_0\sum_i \eta_i^{jk}\sin(K\phi^T_{jk}), \tag{5}$$

where

$$J_0=-\frac{1}{9}\mathrm{Im}\{\det U^+[\vec{q}(0)]\sum_\alpha\sum_i \eta_\alpha^{\beta\gamma}\eta_i^{jk}U^\circ_{\alpha i}U^\circ_{\beta j}U^\circ_{\gamma k}\}, \tag{6}$$

$$\phi^T_{jk}=\phi_j(T)-\phi_k(T).$$

It follows from (5) and (6) that the difference between the probabilities for the transitions $\nu_\beta\to\nu_\gamma$ and $\nu_\gamma\to\nu_\beta$ $(\beta\neq\gamma)$ stems from the *CP* breaking in the lepton sector when there is a complete three-neutrino mixing. Unfortunately, the four quantities $\mathcal{P}_{ee}$, $\mathcal{P}_{\mu\mu}$, $\mathcal{P}_{\tau\tau}$, and $\mathcal{P}$ cannot be measured simultaneously in an experiment with a neutrino beam of a fixed flavor composition. Nevertheless, one might attempt to study *CP*-breaking effects (or *T*-breaking effects) in experiments in the next generation of underground detectors (Super Kamiokande, MACRO, LVD, etc.) through the use of neutrino beams from accelerators and atmospheric neutrinos. This important question requires a separate study.

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[1] Everywhere below, the Greek-letter $(\alpha,\beta,...)$ are used to number to flavors (e,μ,τ), while the Latin-letter indices $(i,j,...)$ take on the values 1, 2, 3.

[2] The symbol η_i^{jk}, which will be used below, is defined in a corresponding way.

[3] We will not reproduce the expression for $U_{\alpha j}$ here, because of its length.

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