Spinor (super-) particle with a commuting index spinor

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An action is proposed for a spinor particle whose configuration space is parametrized by a commuting Weyl spinor along with the vector of space–time coordinates. After quantization, the model describes a massive (massless) particle with a fixed, finite, and otherwise arbitrary spin (helicity). The spinor coordinate plays the role of an index spinor. In the supersymmetric case, the Fermi constraints of massless and massive superparticles with central charges can easily be separated by kind through the use of covariant and irreducible methods. © *1995 American Institute of Physics.*

Studying relativistic quantum formulations of the theory of spinor particles is an important branch of the theory of (super-) p-branes. As we know, a description of spin requires the use of, along with space–time coordinates, some additional "spin" variables, which may be Grassmann variables[1–5] or c-number variables.[6–14] The existing formulations of a spinor particle with c-number spin variables have certain disadvantages, so it is worthwhile to seek new approaches.

In this letter we propose a model of a spinor particle with a c-number Weyl spinor which gives a unified description of the massive and massless cases. In the proper frame, the configuration space of the model is a group manifold of the quantum-mechanical rotation group $SU(2)$ [$E(2)$ for the case of a massless particle]. The space–time coordinates of the model in the natural gauge are Newton–Wigner coordinates, which do not exhibit classical *Zitterbewegung*. In the massive case, the constraints are of the second kind; in the massless case, there is a mixture of constraints of the first and second kinds. The model can be supersymmetrized quite easily. It eliminates the problem of a covariant separation of constraints by kind in the massless case. In the supersymmetric theory, Bose spinors of the model determine a Clifford vacuum of irreducible representations. The result of the quantization is expressed in terms of (anti-) holomorphic functions corresponding to a finite spin, so the condition of microcausality is satisfied.

This model is quite different from existing formulations, using commuting spinors,[6–10] in which the configuration space of the spin is limited to the necessary level, regardless of the choice of proper frame. These spinors can therefore be thought of as generalizations of Ref. 6 of a common type. The spin configuration space in Refs. 8 is the space of a top, so the quantization procedure must be varied (double-valued representations are allowed) in order to find half-integer spins. This situation is also characteristic of the approach of Ref. 7, which uses vector harmonics. The descriptions of the cases $m=0$ and $m \neq 0$ are quite different in all these formulations. The "classical spin" in Refs. 9 and 10 is actually identified with a quantum spin; since the spin operator contains products of noncommuting variables, this approach implies the choice (without a physical basis) of a certain order. The strict fixing of the spin at the classical level restricts the analysis to the effective action of the quantum theory.

A systematic description of a massless particle with a spin using commuting spinors has been offered previously in some identical approaches, a twistor approach[11,12] and a harmonic approach.[13,14] The model under consideration here, with $m=0$, reproduces the twistor formulation of Ref. 12 with some permissible fixing of the gauge.

We write the Lagrangian of this particle in the form

$$L=p\dot{\omega}-\frac{e}{2}(p^2+m^2)-\lambda(\zeta\hat{p}\bar{\zeta}-j), \tag{1}$$

where $\omega_\mu\equiv\dot{\omega}_\mu d\tau\equiv dx_\mu-id\zeta\sigma_\mu\bar{\zeta}+i\zeta\sigma_\mu d\bar{\zeta}$; τ is an evolution parameter; e and λ are Lagrange multipliers; m is the mass; $j\neq 0$ is a dimensionless constant ($\hbar=c=1$); x and p are the 4-coordinate and the 4-energy–momentum; $\hat{p}\equiv p^\mu\sigma_\mu$; and ζ is a commuting Weyl spinor which describes the spin degree of freedom. In a second-order formalism, a physically equivalent Lagrangian is

$$L=\frac{1}{2e}(\dot{\omega}^2-e^2m^2)+g(\zeta\hat{\dot{\omega}}\bar{\zeta}-ej), \tag{2}$$

where $eg=\lambda$.

In the proper frame, action (1) becomes the action of a mechanical system whose configuration space is the group manifold of $SU(2)$ (the quantum-mechanical rotation group); a kinetic term arises because of the use of the form ω, while the spin constraint $\zeta\hat{p}\bar{\zeta}=j$ limits the configuration space to the group manifold. This circumstance is an advantage of our model over the top model. The Borel–Bott theorem is thus clearly manifested in the action which we are proposing here.

The structure of the action is determined by the spinor nature of the spin variables ζ, so we are not surprised, although pleased, to see that by eliminating the spin-constraint term we can make our action the same as the CBS action,[3,15] if we treat the spinor as Grassmann entities. That the result of the "bosonization" of Grassmann vector spin variables is meaningful for a definite spinor particle was pointed out in Ref. 8, among other places.

The spin constraint embodied in the action induces a standard spin constraint which, after quantization, determines the spin of the particle. The constant j is thus a "classical spin," renormalized in the course of the quantization of the order constant. The classical Pauli–Lyubanskiĭ vector $w=(\zeta\hat{p}\bar{\zeta})p-p^2(\zeta\sigma\bar{\zeta})$, which can easily be found by the Noether procedure, has a square $w^2=m^2j^2$ on the surface of constraints. In the case of a massless particle, it is proportional to the momentum: $w=jp$. The value $j=0$ is eliminated, since it renders the spin dynamics trivial.

In the proper-time gauge $xp+\tau m=0$, the equations of motion for the spatial coordinates take the form $m\dot{x}=p$, $p=\text{const}$, so the coordinates in action (1) are Newton–Wigner coordinates (for which there is no *Zitterbewegung*). This circumstance is a definite advantage of the model.

The system of constraints corresponding to action (1) is equivalent to the system

$$p^2+m^2\approx 0, \tag{3}$$

$$S-j\equiv \frac{i}{2}(\zeta p_\zeta-\bar{p}_\zeta\bar{\zeta})-j\approx 0, \tag{4}$$

$$d_\zeta\equiv ip_\zeta-\hat{p}\bar{\zeta}\approx 0,\ \ \bar{d}_\zeta\equiv i\bar{p}_\zeta+\zeta\hat{p}\approx 0. \tag{5}$$

The nonvanishing Poisson brackets of these constraints are

$$\{S,d_\zeta\}=\frac{i}{2}\,d_\zeta,\ \ \{S,\bar{d}_\zeta\}=\frac{i}{2}\,\bar{d}_\zeta,\ \ \{d_\zeta,\bar{d}_\zeta\}=-2i\hat{p}.$$

The mass constraint in (3) and the spin constraint in (4) are thus constraints of the first kind, while the spinor constraints in (5) are of the second kind in the case of a massive particle. In the massless case, in which the matrix $\hat{p}$ is singular, the spinor constraints are a mixture of constraints of the first and second kinds. After we go over to Grassmann spinors and to gauge Poisson brackets, the algebra of spinor constraints becomes the same as the algebra of Fermi constraints of a CBS superparticle. In the massless case the Fermi constraints of a superparticle form an infinitely reducible system, ruling out a covariant quantization, unless auxiliary variables are invoked. In contrast, for a spinor particle [see (1)], the spinor constraints can be separated by kind quite elegantly by using a basis in spinor space formed by the spinors ζ and $\hat{p}\bar{\zeta}$. For the projections of spinor constraints (5) onto these spinors ($\phi\equiv d_\zeta\hat{p}\bar{\zeta},\ \ \chi\equiv\zeta d_\zeta$), we have the following algebra on the surface of constraints (3), (4):

$$\{\chi,\phi\}\approx 0,\ \ \{\bar{\chi},\phi\}\approx 0, \{\phi,\bar{\phi})\approx 0,\ \ \ \{\chi,\bar{\chi}\}\approx -2ij. \tag{6}$$

The Lorentz-invariant projections ϕ and χ and also their complex conjugates are therefore independent constraints of the first and second kinds, respectively.

In the massless case, the constraints of the first and second kinds are sufficient for local elimination of all additional (spin) degrees of freedom from the system of dynamic degrees of freedom. When the mass is nonzero, all the spinor constraints are of the second kind, and we are left with only a single dynamic spin degree of freedom.

Constraints of the second kind of the spinor particle under consideration here form conjugate pairs of commuting constraints, making it natural to choose a Gupta–Bleiler quantization procedure. Correspondingly, we impose the constraints of the first kind and half of the commuting constraints of the second kind on the wave function, so the constraints of the second kind hold on the average. In constructing a quantum algebra of fundamental spinor variables we use a coordinate representation, realizing the operators of the spinor coordinates ζ in terms of multiplication operators, and realizing the conjugate-momentum operators in terms of differentiation operators: $p_\zeta=-i\partial/\partial\zeta$. The particular representation chosen for the space–time coordinates x and the momenta p is unimportant in these questions.

The wave function Ψ is a scalar which depends on the spinor coordinates ζ and $\bar{\zeta}$. If the coordinate representation is selected for the space–time variables, it also depends on the 4-vector x. The constraints of the first kind, mass and spin constraints, lead to the equations

$$(p^2+m^2)\Psi=0, \tag{7}$$

$$(S-s)\Psi=0, \tag{8}$$

where s is the classical spin j, renormalized by the ordering constant. Ordering constants arise upon quantization in expressions which contain products of noncommuting quantities. The requirement that the wave function be single-valued restricts s to half-integer values: $2s \in Z$.

We first focus on a particle with a nonzero mass. Subjecting the wave function to half of the commuting constraints of the second kind, we choose the "holomorphic case":

$$\bar{D}_\zeta \Psi=0, \tag{9}$$

where D_ζ are quantum-mechanical analogs of the constraints d_ζ. We will discuss below the "antiholomorphic" case with the equation $D_\zeta\Psi=0$.

Using expressions for the spin part of the Lorentz generators $M_{\alpha\beta}=\zeta_\alpha\partial/\partial\zeta^\beta$, we find the following expression for the Pauli–Lyubanskiĭ vector $W_{\alpha\dot{\alpha}}=p_{\beta\dot{\alpha}}M_\alpha^\beta-p_{\alpha\dot{\beta}}\bar{M}_{\dot{\alpha}}^{\dot{\beta}}$

$$W^{\alpha\dot{\alpha}}=Sp^{\alpha\dot{\alpha}}+\bar{\zeta}^{\dot{\alpha}}(\bar{D}_\zeta\tilde{p})^\alpha-\zeta^\alpha(\tilde{p}D_\zeta)^{\dot{\alpha}}+2\zeta^\alpha\bar{\zeta}^{\dot{\alpha}}p^2, \tag{10}$$

where $\tilde{p}\equiv p^\mu\tilde{\sigma}_\mu$. Correspondingly, we have

$$\begin{aligned} W^2&=-p^2S(S+1)-p^2(\zeta D_\zeta+1)\bar{\zeta}\bar{D}_\zeta-(\zeta\hat{p}\bar{\zeta})(D_\zeta\hat{p}\bar{D}_\zeta)\\ &=p^2S(1-S)-p^2(\bar{\zeta}\bar{D}_\zeta+1)\zeta D_\zeta-(\zeta\hat{p}\bar{\zeta})\bar{D}_\zeta\tilde{p}D_\zeta, \end{aligned} \tag{11}$$

where the last expression is convenient for an analysis of the antiholomorphic case. Using (7)–(9), we find $W^2=m^2s(s+1)$. For half-integer $s\geqslant 0$, this is the spin for $m^2>0$.

For the field

$$\Phi_+=\exp(+\zeta\hat{p}\bar{\zeta})\Psi, \tag{12}$$

condition (9) becomes the condition for a holomorphic situation, $\partial\Phi_+/\partial\bar{\zeta}=0$, and the spin constraint (8) becomes the condition for homogeneity, $(\zeta\partial/\partial\zeta-2s)\Phi_+=0$. We thus have $s\geqslant 0$, and Φ_+ is a homogeneous polynomial in ζ of degree $2s$. The coefficients of the expansion of Φ_+ in powers of the ζ component of the symmetric undotted spinor of rank $2s$ constitute an ordinary $(2J+1)$-component field which satisfies the Klein–Gordon equation. In the massive case, ζ thus plays the role of an index spinor. The principle of microcausality holds for such fields.

The analysis for the antiholomorphic case is analogous. In this case, however, we have $W^2=-m^2s(1-s)$. The antiholomorphic field $\Phi_-=\exp(-\zeta\hat{p}\bar{\zeta})\Psi$ is homogeneous in $\bar{\zeta}$, of degree $-2s$, and defined everywhere for $s\leqslant 0$. The corresponding ordinary field—a symmetric dotted spinor of rank $2|s|$—has a spin $J=|s|$. The choice of the holomorphic [see (9)] or antiholomorphic condition on the wave function is thus controlled by the sign of s.

In the massless case, the wave function, along with the mass and spin condition (7) and (8), is subject to spinor constraints of the first kind:

$$\bar{\zeta}\tilde{p}D_\zeta\Psi=0, \quad \bar{D}_\zeta\tilde{p}\zeta\Psi=0. \tag{13}$$

For them, mass constraint (7) is a compatibility condition. On Ψ we also impose one of the second-kind constraints, a "holomorphic" one:

$$(\bar{\zeta}\bar{D}_{\zeta}-c+s)\Psi=0. \tag{14}$$

Here $(c-s)$ is a generally complex ordering constant. Using mass condition (7) and spinor constraint (13), we find that the Pauli–Lyubanskiĭ vector in (10) is proportional to the momentum: $W=Sp$. The quantity s is thus the helicity.

For the field

$$\tilde{\Phi}_{+}=(\zeta\hat{p}\bar{\zeta})^{c-s}\Phi_{+}, \tag{15}$$

conditions (14) and the second condition in (13) become the condition for a holomorphic situation with respect to ζ, while spin constraint (8) becomes the condition for homogeneity of degree $2s$. Accordingly, if we assume $\tilde{\Phi}_{+}$ to be defined everywhere with respect to ζ, we conclude that we have $s\geqslant 0$ and that $\tilde{\Phi}_{+}$ is a homogeneous polynomial of degree $2s$ which satisfies the massless Dirac equation in index-free form, because of the first condition in (13) (Ref. 14):

$$\tilde{p}\,\frac{\partial}{\partial\zeta}\,\tilde{\Phi}_{+}=0.$$

In the antiholomorphic case, corresponding to $s\leqslant 0$, condition (14) is replaced by $(\zeta D_{\zeta}-c-s)\Psi=0$, and the field $\tilde{\Phi}_{-}=(\zeta\hat{p}\bar{\zeta})^{c+s}\Phi_{-}$ is an antiholomorphic function with respect to ζ, with a degree of homogeneity $-2s$, which satisfies the Dirac equation

$$\hat{p}\,\frac{\partial}{\partial\zeta}\,\tilde{\Phi}_{-}=0.$$

As in the massive case, ζ is an index spinor. The constant c does not affect the characteristics of the particles of the field, determining the choice of a method for the field description. The field $\Phi_{\pm}$ is the index field of Ref. 14, where it was shown that the microcausality principle holds for this field only if c is a (half-) integer, if $(-)^{2c}=(-)^{2s}$, and if $c\geqslant|s|$.

The model for a spinor particle proposed here can easily be generalized to the super case if we replace the ω form in Lagrangian (1) by

$$\tilde{\omega}_{\mu}=\omega_{\mu}-id\theta\sigma_{\mu}\bar{\theta}+i\theta\sigma_{\mu}d\bar{\theta}, \tag{16}$$

where θ, $\bar{\theta}$ are Grassmann coordinates. The form in (16) is invariant under global nilpotent transformations of spinors $\delta\theta=\epsilon\zeta$, $\delta\zeta=\epsilon\theta$, where ϵ is a real, dimensionless Grassmann parameter. Arbitrary superspins are present in the spectrum of this model. In the Hamiltonian formalism, constraints (3)–(5) are accompanied by the Fermi constraints $d_{\theta}=ip_{\theta}+\hat{p}\bar{\theta}$ and $\bar{d}_{\theta}$, for which we have $\{d_{\theta},\bar{d}_{\theta}\}=2i\hat{p}$. In the massive case, all the spinor constraints are of the second kind; in the massless case, both even and odd spinor constraints are a mixture of constraints of the first and second kinds. In contrast with a CBS superparticle,[15] both boson and fermion spinor constraints can easily be separated by kind in such a model, through the use of a commuting spinor coordinate in the method described above [see (6)]. This result also prevails for the expanded, massive, spin superparticle with central charges, introduced by adding a Wess–Zumino term to the Lagrangian.

The spinor Bose coordinates of the phase space of the model, ζ and p_ζ, have the dimensionality of spinor components of the twistors ω and υ (Ref. 12). In the massless case, it is thus interesting to determine the relationship between the spinor particle under consideration here and the twistor formulation. It turns out that the system of constraints of a massless spinor particle in the positive energy sector is equivalent to the twistor constraint $p_{\alpha\dot{\alpha}}=p_{\zeta\alpha}\bar{p}_{\zeta\dot{\alpha}}/|j|$, which allows a light-like 4-momentum in terms of spinors, a spin constraint with a "classical" helicity j, and the condition $\zeta p_\zeta+\bar{\zeta}\bar{p}_\zeta=0$ (a conformal constraint), which can be thought of as a gauge condition with respect to a twistor constraint. The twistor formulation with a partial fixing of the gauge by virtue of the last condition (a conformal constraint) and with a nonzero classical helicity is introduced by means of a simple replacement of variables: $\omega=|j|^{-1/2}p_\zeta$, $\upsilon=|j|^{1/2}\zeta$. We note the duality of spinor constraints (5) with respect to twistor relations of the type $\omega=\hat{x}\bar{\upsilon}$, $\bar{\omega}=\upsilon\hat{x}$.

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