Exactly integrable model of three-wave mixing in an inhomogeneous nonlinear medium

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A Lagrange identity for conjugate equations is used to construct a Lax representation for an exactly integrable model of three-wave mixing in an inhomogeneous medium with four arbitrary functional parameters: the group velocities and refractive indices of the two primary waves.© *1995 American Institute of Physics.*

The model of three-wave mixing is frequently used to describe the interactions of nearly sinusoidal waves in a nonlinear medium with a quadratic nonlinearity, e.g., in nonlinear optics.[1] This model has been adopted widely because, within the framework of the method of the inverse scattering problem,[2] it is exactly integrable in several cases of interest for applications, e.g., second-harmonic generation.[1] On the other hand, the use of the three-wave-mixing model is usually restricted to homogeneous media, since in an inhomogeneous medium the integrability condition generally does not hold, and solutions with initial conditions in the form of solitons are rapidly destroyed. This is true not only of the model of three-wave mixing but also of other exactly integrable models. There is accordingly interest in the problem of determining the type of inhomogeneous media in which solitons can exist without being destroyed, for each type of exactly integrable models. In the present letter we solve this problem for the model of three-wave mixing. In addition, we resolve the question of the general form of nonlinearity for which the equation of three-wave mixing has a Lax representation and has a soliton solution.

As the initial system of equations we consider the pair of equations

$$\frac{\partial a_i}{\partial t}+v_i(x,t)\frac{\partial a_i}{\partial x}=\sum_{j=1}^{2} w_{ij}a_j. \tag{1}$$

Here a_1 and a_2 are the amplitudes of the two "primary" waves, which are propagating at group velocities v_1 and v_2, respectively, in the medium. The 2×2 matrix $\mathbf{W}$, with the elements w_{ij}, describes the properties of the medium, including the self-effects and interactions of these waves. As a result of the interaction of the primary waves in the nonlinear medium, waves propagating at other group velocities are generally excited. In the simplest case, only a single wave, with amplitude a_3 and group velocity v_3, is excited. The dynamics of this wave is governed by both the properties of the medium and the amplitudes and parameters of the two primary waves. We consider the problem of describing all types of media in which the dynamics of the three interacting waves is such that set of equations (1) has a Lax representation, so waves of the soliton type can exist.

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To solve this problem, we supplement Eqs. (1) with the conjugate system of equations

$$-\frac{\partial\phi_i}{\partial t}-\frac{\partial[v_i(x,t)\phi_i]}{\partial x}=\sum_{j=1}^{2} w_{ji}\phi_j. \tag{2}$$

Here the functions ϕ_i are the conjugates of the functions a_1. Multiplying Eqs. (1) by ϕ_i from the left in a scalar fashion, multiplying Eqs. (2) by a_i from the right, again in a scalar fashion, and taking the difference between the results, we find a generalized conservation law:

$$\frac{\partial}{\partial t}\sum_{i=1}^{2} \phi_i a_i+\frac{\partial}{\partial x}\sum_{i=1}^{2} v_i(x,t)\phi_i a_i=0. \tag{3}$$

This law holds automatically if

$$\frac{\partial\psi}{\partial x}=\sum_{i=1}^{2} \phi_i a_i, \quad \frac{\partial\psi}{\partial t}=-\sum_{i=1}^{2} v_i(x,t)\phi_i a_i, \tag{4}$$

for any function $\psi(x,t)$ which is differentiable and otherwise arbitrary. The appearance of a conservation law for a combination of conjugate equations is a consequence of a generalized Lagrange identity.

We consider the auxiliary vector function $\mathbf{\Psi}=\mathrm{colon}(\psi,\phi_1,\phi_2)$ along with auxiliary 2×2 matrices with elements b_{ij} and a vector c_i such that the following equations hold:

$$\frac{\partial\phi_i}{\partial x}=\sum_{i=1}^{2} b_{ij}\phi_j+c_i\psi. \tag{5}$$

Set of equations (2), (4), and (5) can thus be written as two equations in terms of the vector function $\mathbf{\Psi}$

$$\frac{\partial}{\partial x}\mathbf{\Psi}=\mathbf{U}\mathbf{\Psi}, \quad \frac{\partial}{t}\mathbf{\Psi}=\mathbf{V}\mathbf{\Psi}, \tag{6}$$

where $\mathbf{U}$ and $\mathbf{V}$ are two 3×3 matrices of the form

$$\mathbf{U}(x,t)=\begin{pmatrix} 0 & a_1 & a_2 \\ c_1 & b_{11} & b_{12} \\ c_2 & b_{21} & b_{22} \end{pmatrix}, \quad \mathbf{V}(x,t)=\begin{pmatrix} 0 & v_1 a_1 & v_2 a_2 \\ v_1 c_1 & d_{11} & d_{12} \\ v_2 c_2 & d_{21} & d_{22} \end{pmatrix}, \tag{7}$$

where $d_{ij}=w_{ji}+v_i b_{ij}+v_{i,x}\delta_{ij}$, $i,j=1,2$. Set of equations (2), (4) contains the original system of equations, (1), so auxiliary relations (5), which do not impose any additional restrictions on the functions ϕ_i, ψ, and a_i, have the consequence that the entire set of equations (6) contains the original system of equations, (1). It follows that the condition for the compatibility of system (6), which can be written as the Zakharov–Shabat "zero curvature" condition,[2]

$$\mathbf{U}_t-\mathbf{V}_x+[\mathbf{U},\mathbf{V}]=0, \tag{8}$$

(where [,] is the ordinary matrix commutator), is the initial equation along with an additional set of equations for the auxiliary functions b_{ij} and c_i. The form of these equations can be determined through an explicit calculation of conditions (8). Equations (6)–(8) thus form an analog of the Lax representation of the original system of equations.

If Eqs. (6)–(8) are to constitute an exact Lax representation, matrices (7) must contain an arbitrary complex parameter λ which transforms system (6) into a nontrivial system of two spectral problems with λ as a spectral parameter, on which the unknown functions a_i of the original equation do not depend. These conditions hold if we assume that only the auxiliary functions b_{ij} and c_i depend on λ. The simplest case of such a dependence corresponds to the assumption

$$b_{11}=P_1(x,t)\lambda+R_{1x}(x,t),\quad b_{22}=P_2(x,t)\lambda+R_{2x}(x,t) \tag{9}$$

under the condition that b_{12}, b_{21}, and c_i are independent of λ. Substituting (9) into (8), making the replacement $a_k\rightarrow ia_k$, and using the additional reduction

$$a_k^*=ic_k,\quad b_{12}=b_{21}^*=ia_3, \tag{10}$$

we find the following system of complex equations:

$$\begin{aligned}
&\frac{\partial a_1}{\partial t}+\frac{\partial[v_1(x,t)a_1]}{\partial x}+iN_1(x,t)a_1=iQ_1(x,t)a_2a_3^*,\\
&\frac{\partial a_2}{\partial t}+\frac{\partial[v_2(x,t)a_2]}{\partial x}+iN_2(x,t)a_2=iQ_2(x,t)a_1a_3,\\
&\frac{\partial a_3}{\partial t}+\frac{\partial[v_3(x,t)a_3]}{\partial x}+iN_3(x,t)a_3=iQ_3(x,t)a_2a_1^*.
\end{aligned} \tag{11}$$

These equations have the form of generalized three-wave-mixing equations. Here we are using the following notation:

$$\begin{gathered}
v_3(x,t)=v_1(x,t)+Q_2(x,t)=v_2(x,t)+Q_1(x,t),\\
Q_2(x,t)=(v_1-v_2)P_2/(P_1-P_2)=(v_3-v_1),\\
Q_1(x,t)=(v_1-v_2)P_1/(P_1-P_2)=(v_3-v_2),\\
Q_3(x,t)=Q_1-Q_2=(v_1-v_2),\\
N_1(x,t)=\mathbf{D}_1R_1\equiv R_{1t}+v_1R_{1x},\\
N_2(x,t)=\mathbf{D}_2R_2\equiv R_{2t}+v_2R_{2x},\\
N_3(x,t)=(v_1-v_2)[R_{2x}+Q_2(R_{2x}-R_{1x})]+\mathbf{D}_2R_2-\mathbf{D}_1R_1.
\end{gathered} \tag{12}$$

In an optical interpretation of this model, the functions N_1, N_2, and N_3 correspond to the refractive indices of waves in a medium; in the case at hand, these indices are functions of the coordinates and the time. The functions Q_1, Q_2, and Q_3 describe the inhomogeneity of the nonlinear properties of the medium. The functions $R_1(x,t)$ and $R_2(x,t)$ in (12) are completely arbitrary, while the functions $P_1(x,t)$ and $P_2(x,t)$ are related to the functions $v_1(x,t)$ and $v_2(x,t)$ (the group velocities of the two primary waves) by the two equations

$$\frac{\partial P_1}{\partial t}+\frac{\partial}{\partial x}[v_1(x,t)P_1]=0,\quad \frac{\partial P_2}{\partial t}+\frac{\partial}{\partial x}[v_2(x,t)P_2]=0. \tag{13}$$

Accordingly, any two of the four functions $P_1(x,t)$, $P_2(x,t)$, $v_1(x,t)$, $v_2(x,t)$ are also arbitrary.

In this notation, the functions w_{ij} take the form

$$w_{12}=Q_1a_3^*,\quad w_{21}=Q_1a_3,\quad w_{11}=-\mathbf{D}_1R_1-v_{1x},\quad w_{22}=-\mathbf{D}_2R_2-v_{2x}.$$

The entire set of relations determines the form of the matrices **U** and **V** of representation (6)–(8) for which the latter is a true Lax representation. As result, it is possible to use the method of the inverse scattering problem to construct exact solutions of these equations.

However, it is generally difficult to apply this method directly to the system of equations above. We can put system (11) in a simpler form by using the transformation

$$(x,t)\to[\theta_1(x,t),\theta_2(x,t)], \tag{14}$$

$$a_k(x,t)\to A_k[\theta_1(x,t),\theta_2(x,t)],\quad k=1,2,3,$$

where

$$A_1[\theta_1(x,t),\theta_2(x,t)]=\frac{a_1}{P_1}\,e^{iR_1},\quad A_2[\theta_1(x,t),\theta_2(x,t)]=\frac{a_2}{P_2}\,e^{iR_2}, \tag{15}$$

$$A_3[\theta_1(x,t),\theta_2(x,t)]=\frac{a_3}{P_1-P_2}\,e^{i(R_2-R_1)}.$$

As new independent variables here we have selected θ_1 and θ_2, which are related to P_1 and P_2 by

$$\partial_x\theta_1=P_1,\quad \partial_x\theta_2=P_2$$

and which therefore satisfy the equations

$$\partial_x\theta_1+v_1\partial_x\theta_1=0,\quad \partial_t\theta_2+v_2\partial_x\theta_2=0.$$

As a result, system (11) becomes

$$\frac{\partial A_1}{\partial\theta_2}=iA_2A_3^*,\quad \frac{\partial A_2}{\partial\theta_1}=iA_1A_3,\quad \frac{\partial A_3}{\partial\theta_1}+\frac{\partial A_3}{\partial\theta_2}=iA_1A_2^*. \tag{16}$$

This system is one version of the standard system of equations for three-wave mixing, corresponding to the case $v_1=\text{const}$; $v_2=$ const; R_1, $R_2=0$. We can thus first construct soliton solutions directly for system (16) (simple solutions are discussed in Ref. 1, among other places); then, with the help of the transformation which is the inverse of (14), (15), we can find the amplitudes of the initial equation in the inhomogeneous medium.

Using this system, we can analyze the behavior of three-wave-mixing solitons in an inhomogeneous medium whose properties are determined by the four arbitrary real functions N_1, N_2, v_1, and v_2. Since these functions are arbitrary, they may contain a dependence on the amplitudes of the waves in the medium, so the medium may have nonlinear properties which are not the same as those usually considered in the exactly integrable model of three-wave mixing. The integrability of these equations by the method of the inverse scattering problem is preserved. We assume $R_i = R_i(x,t,a_1,a_2,a_3)$, $v_i = v_i(x,t)$, $i = 1,2$, with an arbitrary functional dependence of v_i on x,t and also of R_i on a_k. We further assume that A_1, A_2, and A_3 are solutions of Eqs. (16), while P_1, P_2, and $P_3 = P_2 - P_1$ are solutions of Eqs. (13). According to (15), we then have $|a_k| = |A_k| P_k$, and the functions $\arg(a_k)$ are found from the solution of a system of (generally transcendental) algebraic equations:

$$\arg(A_1) = \arg(a_1) + R_1(x,t,a_1,a_2,a_3),$$
$$\arg(A_2) = \arg(a_2) + R_2(x,t,a_1,a_2,a_3),$$
$$\arg(A_3) = \arg(a_3) + R_2(x,t,a_1,a_2,a_3) - R_2(x,t,a_1,a_2,a_3). \quad (17)$$

In the case $v_i = v_i(x,t,a_1,a_2,a_3)$, i.e., when there is a dependence on the amplitudes a_k, the latter system of algebraic equations should be solved jointly with Eqs. (13). This requirement seriously complicates the problem of analyzing the soliton dynamics.

The simplest nontrivial use of the resulting representation to study physical problems, for which we can find solutions of system (17) in a fairly simple way, corresponds to a functional dependence $R_i = R_i(x,t|a_1|,|a_2|,|a_3|)$, i.e., to a dependence on only the absolute values of the amplitudes. As an example of such a system we might cite a generalized analog of the massive Thirring model.[3] We set

$$N_1(x,t) \equiv \mathbf{D}_1 R_1 = n_1(a_1,a_2,a_3),$$
$$N_2(x,t) \equiv \mathbf{D}_2 R_2 = n_2(a_1,a_2,a_3).$$

Choosing $n_1 = |a_2|^2 + m_1$ and $n_2 = |a_1|^2 + m_2$ under the conditions m_1, $m_2 = \text{const}$; $v_1 = v_2 = \text{const}$; and $a_3 \equiv 0$, we then find a system which is equivalent to the massive Thirring model and which can be integrated by the method of the inverse scattering problem.[3] We might add that in the case $R_i = R_i(x,t,|a_1|,|a_2|,|a_3|)$ a solution of Eqs. (13) can be carried out without the use of (17), even if v_1 and v_2 depend on the amplitudes a_k. Models of this type were discussed in Ref. 1; in several cases they can be analyzed completely by this approach.

We note in conclusion that the approach discussed here can be extended to the case of the interaction of N waves in an inhomogeneous medium. To do this, we need to consider the original problem, (1), as the problem of the propagation of n "primary" waves in an inhomogeneous nonlinear medium. In this case the final model would correspond to the problem of the interaction of $N = n(n+1)/2$ waves, and a Lax representation would be realized in terms of $(n+1)\times(n+1)$ matrices and would contain $2n$ arbitrary functional parameters (n group velocities and n refractive indices for the primary waves). Other models, e.g., one with a quadratic dispersion relation, could be adopted as the "primary" model for wave propagation in the medium. The equations of these models could also be incorporated in the scheme proposed above for constructing Lax representations on the basis of a Lagrange identity.

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[1] A. P. Sukhorukov, *Nonlinear Wave Interaction Processes in Optics and Radiophysics* [in Russian] (Nauka, Moscow, 1988).

[2] V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskiĭ, *Soliton Theory: Method of the Inverse Problem* [in Russian] (Nauka, Moscow, 1980).
[3] E. A. Kuznetsov and A. B. Mikhaĭlov, Teor. Mat. Fiz. **30**, 193 (1977).

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