Stability of power-law weak-turbulence distributions in superfluid helium

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The stability of power-law spectra of acoustic turbulence in superfluid helium is analyzed. Such spectra are shown to be unstable with respect to isotropic perturbations and also with respect to perturbations which have the angular distribution of the first spherical harmonic. Corrections to the distributions for the onset of this instability are derived. © *1995 American Institute of Physics.*

Power-law distributions of fluctuating properties can arise in superfluid helium in a regime of acoustic turbulence.[1,2] Such distributions are exact, steady-state, nonequilibrium solutions of the corresponding kinetic equations for waves.[3] They are of the form

$$N_{\mathbf{k}},\ n_{\mathbf{k}} \propto k^{s}, \tag{1}$$

where $N_{\mathbf{k}}$ and $n_{\mathbf{k}}$ are the occupation numbers of modes with wave vector $\mathbf{k}$ for first and second sound, and s takes on the value $-9/2$ or -4. The spectrum with the first of these exponents forms at large values of k (Ref. 2), while that with the second forms at small values of k (Ref. 4). In this letter we examine the stability of turbulent distributions of this sort with respect to small and otherwise arbitrary perturbations of the initial conditions and also with respect to weak external sources.

A direct estimate of the contributions of various nonlinear interactions to the collision integrals of the kinetic equations shows that the decay of a first-sound wave into two second-sound waves is the predominant process in the roton region of helium temperatures ($T \geqslant 1$ K). If we incorporate that process alone, the kinetic equations become[1]

$$\frac{\partial}{\partial t} N_{\mathbf{k}} = \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}(n_1 n_2 - N_{\mathbf{k}} n_1 - N_{\mathbf{k}} n_2) + \Gamma_1(\mathbf{k},t),$$

$$\frac{\partial}{\partial t} n_{\mathbf{k}} = -\int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}(n_{\mathbf{k}} n_1 - N_2 n_{\mathbf{k}} - N_2 n_1) - \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}}(n_{\mathbf{k}} n_2 - N_1 n_{\mathbf{k}} - N_1 n_2) + \Gamma_2(\mathbf{k},t), \tag{2}$$

where

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \pi |V_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2}|^2 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\Omega-\omega_1-\omega_2);$$

$V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ is the amplitude of this interaction, which is a homogeneous function of k with an exponent of 3/2 (Ref. 5); $\Omega=c_1k$; $\omega_i=c_2k_i$ $(i=1,2)$; c_1 and c_2 are the velocities of first and second sound; $\Gamma_1(k,t)$, and $\Gamma_2(k,t)$ are weak external sources; and for brevity we are using the notation $N_i=N_{\mathbf{k}_i}$, $n_i-n_{\mathbf{k}_i}$. In the inertial interval of k space, the equalities $\Gamma_1=\Gamma_2=0$ hold.

In the stability analysis below we assume that the deviations $\delta N_{\mathbf{k}}(t)$ and $\delta n_{\mathbf{k}}(t)$ from steady-state distribution (1) are small in comparison with the quantities $N_{\mathbf{k}}$ and $n_{\mathbf{k}}$ themselves (i.e., we assume $\delta N_{\mathbf{k}}/N_{\mathbf{k}},\ \delta n_{\mathbf{k}}/n_{\mathbf{k}}\ll 1$). We can therefore linearize Eqs. (2) with respect to these deviations. Expanding them in spherical harmonics Y_{lm},

$$\delta N_{\mathbf{k}}(t)=\sum_{l,m}\ \delta N_{lm}(k,t)Y_{lm}(\boldsymbol{\nu}),$$

$$\delta n_{\mathbf{k}}(t)=\sum_{l,m}\ \delta n_{lm}(k,t)Y_{lm}(\boldsymbol{\nu}), \tag{3}$$

where $\boldsymbol{\nu}=\mathbf{k}/|\mathbf{k}|$ is a unit vector, we find a system of linear integrodifferential equations in the functions $\delta N_{lm}(k,t)$ and $\delta n_{lm}(k,t)$. By virtue of the initial isotropy of the medium, these equations are decoupled with respect to the index l, and the dependence on m drops out altogether. We will analyze the stability on the basis of the resulting system of equations.

The theory has a small parameter: the ratio of the velocities of first and second sound, $\gamma=c_2/c_1$. (According to experimental data, in the temperature range under consideration here the value of γ is $\gamma\approx 10^{-1}$; Ref. 6, for example.) We can thus simplify the analysis substantially and use the adiabatic approximation. Specifically, the time scale for a change in the second-sound distribution differs by a factor on the order of γ^3 from the corresponding time scale for second sound. In the adiabatic approximation, the functions $\delta N_{lm}(k,t)$, which play the role of fast variables, are eliminated. The equations which are left after this elimination determine the slow evolution of perturbations $\delta n_{lm}(k,t)$. The form of the functions $\delta N_{lm}(k,t)$ can be determined unambiguously for each instant. The stability of these equations can be analyzed by the method of Ref. 7.

Using the Mellin transformation

$$G_{lm}(z,t)=\int_0^\infty dk k^{z-s-1}\delta n_{lm}(k,t), \tag{4}$$

we find equations for $G_{lm}(z,t)$ (we are omitting the indices l and m):

$$\dot{G}(z-h,t)=W(z)G(z,t)+\Psi(z-h,t). \tag{5}$$

Here $\Psi(z,t)$ is the Mellin transform of the external source, the constant h is $h=s+5$, where s is the exponent in the distribution under consideration, (1), and the function $W(z)$ is equal to the determinant of the second-rank matrix $\hat{W}(z)$ with the matrix elements

$$w_{11}=-\int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}(n_1+n_2)k^{-h},$$

$$w_{12}(z)=2\int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} P_l(\boldsymbol{\nu}\times\boldsymbol{\nu}_2)(n_1-N_{\mathbf{k}})n_2 N_{\mathbf{k}}^{-1} k^{z-h} k_2^{-z},$$

$$w_{21}(z)=\int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}} P_l(\boldsymbol{\nu}\times\boldsymbol{\nu}_1)(n_{\mathbf{k}}+n_2)N_1 n_{\mathbf{k}}^{-1} k^{z-h} k_2^{-z}, \tag{6}$$

$$w_{22}(z)=-2\int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}} P_l(\boldsymbol{\nu}\times\boldsymbol{\nu}_2)(n_{\mathbf{k}}-N_1)n_2 n_{\mathbf{k}}^{-1} k^{z-h} k_2^{-z}$$
$$-2\int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}}(n_2-N_1)k^{-h}.$$

Here $P_l(x)$ is the Legendre polynomial. The functions $w_{ab}(z)$, where $a,b=1,2$, have a zero degree of homogeneity with respect to k.

The stability of Eq. (5) is determined by the Zakharov–Balk condition.[8] For this purpose we define the rotation function $\kappa(\zeta)$ as the increment in the complex argument of the function $W(z)$ as z goes from $\zeta-i\infty$ to $\zeta+i\infty$, divided by 2π. The function $\kappa(\zeta)$ takes on only integer values.

Power-law distribution (1) is stable with respect to initial perturbations and with respect to an external source if and only if there exists an interval of zero rotation Δ_0, $\{\Delta_0=(\zeta_-,\zeta_+)|\kappa(\zeta)=0,\zeta\in\Delta_0\}$ and $\kappa(0)=0$. A sufficient condition for instability is $W(0)>0$. The zeros z_p of the function $W(z)$ determine the steady-state corrections to the distribution which form:

$$\delta N_{\mathbf{k}},\ \delta n_{\mathbf{k}}=\mathrm{const} k^{s-z_p} Y_{lm}(\boldsymbol{\nu}).$$

The integrals in (6) which appear in the definition of $W(z)$ cannot be evaluated in general form for arbitrary values of z. We have tested the stability with respect to arbitrary perturbations through a numerical estimate of the quantities $\kappa(0)$ for various values of l. The functions w_{ab}, where $a,b=1,2$, were put in a convenient form: A Zakharov transformation[7] was carried out in the integrals which appear in w_{21} and w_{22}, and the δ-functions were eliminated through an integration over $\mathbf{k}_1$ and over the absolute value $|\mathbf{k}_2|$. An expression for the amplitude $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ was taken from Ref. 5.

We calculated rotation numbers $\kappa(0)$ for $l\leq 20$; we also found the zeros z_p of the function $W(z)$: $W(z_p)=0$.

In the case $l=0$ the origin of coordinates lies on the boundary of the zero-rotation interval. This situation corresponds to the case in which the amplitudes of distribution (1) are not fixed by Eqs. (2).

As a result of a numerical estimate of the positions of the zeros z_p of the function $W(z)$, which satisfy the Falkovich condition (that the sign of the flux of the integral of motion of the linearized equation which results from a correction $\delta n(k)$ be the same as the sign of the flux of the integral of the background distribution[9]), we found the isotropic steady-state corrections:

$$\delta N_k,\ \delta n_k \propto \begin{cases} k^{-13.9}, & k\to 0,\\ k^{-1.53}, & k\to\infty. \end{cases} \tag{7}$$

In the case $l=1$ the results are different for different values of s. For $s=-4$, the calculations yield $\kappa(0)=2$, which corresponds to an instability of distribution (1) at small values of k. Estimating z_p, we find a steady-state correction which satisfies the Falkovich condition:

$$\delta N_{\mathbf{k}},\ \delta n_{\mathbf{k}}=\text{const}\ k^{-5}\cos\theta,\quad k\rightarrow 0. \tag{8}$$

The spatial direction from which the polar angle θ is reckoned is determined by the anisotropy of the source.

For the value $s=-9/2$ we calculated the quantity $\kappa(\epsilon)$ for $\epsilon>0$, $\epsilon\rightarrow 0$. In this case, the introduction of a nonzero value of ϵ is explained in the following way. The linear model dispersion relation which we have adopted, $\Omega_{\mathbf{k}}$, $\omega_{\mathbf{k}}\propto k$, leads to the equality $W(0)=0$, so the quantity $\kappa(0)$ is not determined. When the decay nature of the dispersion in real helium[10,11] is taken into account by means of (for example) the substitution[7] $\Omega_{\mathbf{k}}=c_1 k^{1+\epsilon}$, $\omega_{\mathbf{k}}=c_2 k^{1+\epsilon}$ $(\epsilon\rightarrow +0)$, we find that the zeros of the function $W(z)$ shift by a small negative amount on the order of $-\epsilon$. This is an important factor in the case under consideration here, since the origin of coordinates, $z=0$, now lies inside the zero-rotation interval of the function $W(z)$. The same factor can evidently be taken into account by calculating the quantity $\kappa(\epsilon)$; a numerical estimate yields a zero value for this quantity. This result indicates that turbulent distribution (1) is stable in the region $k\rightarrow\infty$ with respect to perturbations with $l=1$. For $l=2-20$, an estimate yields $\kappa(0)=0$ for both values of the exponent s.

Distribution (1) is therefore unstable with respect to the formation of isotropic corrections and of corrections having the form of the first spherical harmonic. With respect to perturbations with other values of l, distribution (1) is stable. In general, the stability of power-law distributions with respect to perturbations with large values of l is guaranteed by the general theory.[8] It can thus be concluded that the instabilities which have been found are the only ones. The steady-state corrections which form are given by relations (7) and (8).

In connection with this instability and the growth of the deviations from the background spectrum in the limits $k\rightarrow 0$ and $k\rightarrow\infty$, one might ask about the validity of the linearized theory. Analysis of the positions of the zeros z_p shows that the instability that arises is of an interval type. By this we mean that the corrections to the spectrum are small over length scales close to the length scale of the pump, k_p, and they grow in a power-law fashion toward the ends of the inertial interval.[8] The use of linearized equations in this case is justified by the fact that in a real system the inertial interval (k_0,k_d) is finite $(0<k_0<k_p<k_d<\infty)$. The distribution that forms is assumed to be approximately equal to the sum of (1), (7), and (8) inside the inertial interval and to deviate greatly from this sum at $k\gg k_d$ and $k\ll k_0$. If the linear approximation is to be valid, it is thus necessary that the inequalities $\delta N_{\mathbf{k}}\ll N_k$ and $\delta n_{\mathbf{k}}\ll n_k$ hold only inside the inertial interval. When these conditions do hold, the results are correct for all $k\in(k_0,k_d)$.

In the opposite case, the interval in k space, in which a distribution of this sort forms, is smaller than the inertial interval. It is bounded by the conditions $\delta N_{\mathbf{k}}\sim N_k$ and $\delta n_{\mathbf{k}}\sim n_k$, and it always contains the pumping region (by virtue of the interval nature of the instability). For wave vectors k which are sufficiently close to k_p, the observed distribution may be slightly different from unperturbed spectrum (1).

An interesting feature of this system is the instability of the power-law turbulent distribution in (1) with respect to isotropic perturbations. This case is generally atypical of real physical systems. The reason is that the nonlinear decay of the acoustic wave into two waves with a smaller sound velocity is a local process in k space. Because of this local nature, the Mellin function $W(z)$ is meromorphic over the entire plane of the complex variable z. All the zeros z_p thus lie in a region in which Mellin transformation (4) is invertible, so corresponding steady-state corrections may arise. In the case of a nonlinear interaction of a different type, the zeros of the corresponding function for $l=0$ usually lie outside this region, so no isotropic power-law corrections to a distribution of the type in (1) can arise.[7]

The distribution that actually forms, in which the steady-state corrections are taken into account, depends on only the absolute value of the vector $\mathbf{k}$ in the limit $k \to \infty$. It is anisotropic in the limit $k \to 0$. The physical reason for this result is the formation of a slight momentum flux [which is also an integral of motion of Eqs. (2)] in the long-wave part of k space, against the background of the main distribution. An angular dependence of the type in (8) at small values of k could in principle be observed experimentally.

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