Five-lepton modes for the decay of μ and τ mesons

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(Submitted 23 April 1993)
Pis'ma Zh. Eksp. Teor. Fiz. **57**, No. 12, 746–749 (25 June 1993)

Differential widths with respect to the fraction of the energy carried by the charged leptons are calculated for $\mu, \tau \to 3l \cdot 2\nu$ decays. The case in which a "soft" pair is produced is singled out for special study. One-particle distributions are analyzed. The background for a search for the neutrinoless decay $\mu \to 3e$ and for the muonium-antimuonium transition $\mu^+e^- \to \mu^-e^+$ is estimated. The results are used to generate estimates of the spectra of five-lepton decays of the τ meson in the doubly logarithmic approximation.

The decay

$$\mu^+ \to e^+\nu_e\bar{\nu}_\mu e^+e^- \tag{1}$$

was observed first in 1959 and later in 1975.[1] A numerical simulation of this decay was carried out in Ref. 2. In the experiments of Ref. 3, 7874 events of this decay were recorded. They were analyzed in Ref. 4 from the standpoint of the restrictions imposed on the structure of the weak current. Monte Carlo calculations based on the decay matrix element within the framework of the Standard Model agree with the experimental value of the relative width,[5]

$$R = \frac{\Gamma(\mu \to 3e2\nu)}{\Gamma_{\text{tot}}} = (3.4 \pm 0.4) \times 10^{-5}. \tag{2}$$

This decay constitutes a background in studies of muon-antimuon conversion, $\mu^+e^- \to \mu^-e^+$ (Ref. 6), and of the neutrinoless decay $\mu^+ \to e^+e^+e^-$ (Ref. 7). With c/τ factories coming on line,[8] this decay can be used to refine the Standard Model and to measure possible deviations from it as a normalization process.

In the present letter we report analytic calculations of the spectra of charged leptons for the decay of μ and τ mesons.

The spectral distribution in terms of the relative energies of the charged leptons is (the details of the calculations are given in Ref. 9)

$$\frac{d^3\Gamma}{\Gamma_0\, dx_1\, dx_2\, dy} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 (1+P_{12})(L^2 f_1 + L f_2 + f_3), \tag{3}$$

where the operator P_{12} interchanges x_1 and x_2,

$$x_{1,2}\frac{2\epsilon^+_{1,2}}{m_\mu}, y = \frac{2\epsilon^-}{m_\mu}, \quad x_1 + x_2 + y = \Delta \leqslant 1, \tag{4}$$

$$\sigma \leqslant x_1, x_2, y \leqslant 1 - \frac{4m_e}{m_\mu}, \quad L = ln\frac{m_\mu^2}{m_e^2} \simeq 11. \tag{5}$$

The quantity $m_\mu \sigma/2$ is the minimum lepton energy which can be detected experimentally. The leading (doubly logarithmic) contribution to spectrum (3) comes from the kinematic region in which the pair which is produced moves along with the third charged lepton in the direction opposite the momenta of the neutrinos, carrying off half the energy:

$$f_1 = \frac{1}{2}(2\Delta - 3)\frac{x_1 x_2 y}{(x_1+y)^2}\left[\frac{\Delta}{(x_1+y)^2} + \frac{x_1 y - x_2\Delta - \frac{1}{2}(x_1+y)^2}{2x_1 x_2 y}\right]\Theta(1 - x_1 - x_2 - y)\,. \tag{6}$$

The expression for the function f_2 is quite unwieldy; it is given in Ref. 9. For values $x_1, x_2, y \sim 1$, these functions are on the order of unity. They are smooth functions. They increase as their arguments approach the lower boundary ($x_i, y \to x_{\min}, y_{\min}$). We believe that the function f_3 has corresponding properties, so ignoring it will not introduce any uncertainties greater than 1% in the spectral distributions or the total width of this decay channel.

Integrating over x_1 and x_2, we find from (3)

$$\frac{d\Gamma^{\mu\to ee\bar{e}\nu\bar{\nu}}}{\Gamma_0\, dy} = \left(\frac{\alpha}{\pi}\right)^2 [L^2\Phi_1(y) + L\Phi_2(y)] \equiv \left(\frac{\alpha}{\pi}\right)^2 \Phi(y), \tag{7}$$

where

$$\Phi_1(y) = \frac{1}{6y} + \frac{17}{36} + \frac{3}{4}y - \frac{7}{6}y^2 - \frac{2}{9}y^3 + \left(\frac{5}{12} + y + y^2\right)\ln y,$$

$$\begin{aligned}
\Phi_2(y) = {} & \left(\int_y^1 \frac{dx}{x}\ln(1+x)\right)\left[\frac{272}{3}y^3 + 136y^2 + 32y - \frac{40}{3}\right] + \left(\int_y^1 \frac{dx}{x}\ln(1-x)\right) \\
& \times\left[-2y^2 - 2y - \frac{5}{6}\right] + (16^3 + 24y^2)\int_y^1 \frac{dt_1}{t_1}\int_0^{1-t_1}\frac{dx_2}{x_2+y}\ln\left(\frac{(x_2+y)t_1}{\Delta y}\right) \\
& + \left[-\frac{4}{9}y^3 - \frac{7}{3}y^2 + \frac{3}{2}y\frac{17}{18} + \frac{1}{3y}\right]\ln(1-y) + \left[-\frac{371}{6}y^2 - \frac{277}{12}y + \frac{34}{3} + \frac{2}{3y}\right]\ln(y) \\
& + \left[-\frac{136}{3}y^3 - 50y^2 - 5y + \frac{55}{12}\right]\ln^2(y) \\
& + \left[\frac{272}{3}y^3 + 136y^2 + 32y - \frac{40}{3}\right]\ln(y)\ln(1+y) \\
& + \frac{877}{54} - \frac{1949}{36}y - \frac{1}{6y} - \frac{631}{36}y^2 + \frac{1501}{27}y^3,
\end{aligned} \tag{8}$$

$$t_1 \equiv x_1 + y. \tag{9}$$

The behavior of the function $\Phi(y)$ as $y \to 1$ must be analyzed in order to evaluate the background for the muonium-antimuonium transition $(\mu^+ e^- \to {}^- e^+)$.[6] We found the following expression for $\Phi(y)$ in this region:

$$\Phi(y)|_{y\to 1} \approx (1-y)^2 \left[\frac{1}{8} L^2 - \left(\frac{13}{12} - \frac{1}{4} \ln(1-y) \right) L \right]. \tag{10}$$

This expression differs from the estimate given in Ref. 7 in that the coefficient of the first power of L is different.

From distribution (2) we can find a distribution with respect to Δ, i.e., the fraction of the energy which is carried off by charged leptons $(\Delta = x_1 + x_2 + y)$. For the region $1 \leqslant \Delta \leqslant 2$ it is

$$\frac{d\Gamma^{\mu \to ee\bar{e}\nu\bar{\nu}}}{\Gamma_0 \, d\Delta} = \left(\frac{\alpha}{\pi} \right)^2 L \Psi(\Delta), \tag{11}$$

where

$$\Psi(\Delta) = \frac{2}{3} \Delta^2 (3 - 2\Delta) \left[\frac{1}{2} \ln^2(1-\Delta) + 2 \int_\Delta^2 \frac{dy}{y} \ln(1-y) \right]$$
$$+ \ln(1-\Delta) \left(\frac{5}{18} + \frac{2}{3} \Delta + \frac{2}{3} \Delta^2 - \frac{4}{9} \Delta^3 \right)$$
$$+ (2-\Delta) \left(\frac{52}{45} + \frac{211}{60} \Delta - \frac{491}{180} \Delta^2 + \frac{517}{1080} \Delta^3 - \frac{17}{540} \Delta^4 + \frac{1}{270} \Delta^5 \right).$$

As $\Delta \to 2$ we find

$$\Psi(\Delta) \approx \frac{13}{36} (2-\Delta)^2. \tag{12}$$

The distributions can be used to evaluate the background for the decay $\mu \to e^+ e^+ e^-$.[1,3]

Spectrum (3) must be modified in the region $y \sim 2m_e/m_\mu \ll 1$. Using the soft-pair approximation,[10] we find

$$\frac{d\Gamma}{d\Gamma_0} = \frac{d\Gamma}{d\Gamma_0} + \frac{2}{3} \left(\frac{\alpha}{\pi} \right)^2 \left\{ \frac{1}{3} \left(\frac{1}{2} L + \ln \eta \right)^3 - \frac{4}{3} \left(\frac{1}{2} L + \ln \eta \right)^2 + \left(-\frac{\pi^2}{6} + \frac{61}{18} \right) \left(\frac{1}{2} L + \ln \eta \right) \right\},$$

where $\eta = 2\Delta\epsilon/m_\mu$, and $\Delta\epsilon$ is the total energy of the components of a soft pair. We assume

$$\frac{2m_e}{m_\mu} \ll \eta \ll 1.$$

Integrating expression (3)under the condition that the fraction of the energy carried by the pair, t, is greater than η, we find

$$\frac{\Gamma^{hard}}{\Gamma_0}=\int_\eta^1 \frac{d\Gamma^{hard}}{\Gamma_0 dt}\,dt$$

$$=\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 L\left(-\frac{2}{3}\ln^2\eta-\frac{1}{3}\,L\ln\eta+\frac{16}{9}\ln\eta-\frac{7}{18}-12\zeta_2+8\zeta_3+\frac{3565}{216}\right), \quad (14)$$

$$\zeta_2=\sum_1^\infty n^{-2}=\frac{\pi^2}{6},\quad \zeta_3=\sum_1^\infty n^{-3}\approx 1.202.$$

The complete expression for the contribution from soft and hard pairs to the total width is

$$\frac{\Gamma^{\rm hard}+\Gamma^{\rm soft}}{\Gamma_0}=\left(\frac{\alpha}{\pi}\right)^2\left\{\frac{1}{36}\,L^3-\frac{5}{12}\,L^2+L\left(\frac{1351}{144}+4\zeta_3-\frac{19}{3}\,\zeta_2\right)+O(1)\right\}. \quad (15)$$

This expression is independent of the parameter η; this point is a test of the self-consistency of the calculation. Let us examine the contribution to the total width of the muon decay which comes from the interference of the Born amplitude ($\mu\to e\nu\bar{\nu}$) with the amplitude incorporating a two-loop correction with a photon polarization operator. We restrict the discussion to the contributions of electrons and μ mesons to the polarization operator, since the contributions of τ leptons and hadrons do not contain L:[9]

$$\frac{d\Gamma^{\rm virt}}{d\Gamma_0}=\left(\frac{\alpha}{\pi}\right)^2\left\{-\frac{1}{36}\,L^3+\frac{3}{8}\,L^2+\frac{1}{6}\left(\ln^2\frac{M_W^2}{m_\mu^2}+L\ln\frac{M_W^2}{m_\mu^2}\right)\right\}, \quad (16)$$

$$L=\ln\left(\frac{m_\mu^2}{m_e^2}\right)\approx 10.7,\quad \ln\left(\frac{M_W}{m_\mu}\right)\approx 6.6.$$

For the complete contribution from virtual pairs, (16), and real pairs, (14), we have

$$\frac{\Gamma^{\rm hard}+\Gamma^{\rm soft}+\Gamma^{\rm virt}}{\Gamma_0}=\left(\frac{\alpha}{\pi}\right)^2\left\{-\frac{1}{24}\,L^2+\frac{1}{6}\left(\ln^2\frac{M_W^2}{m_\mu^2}+L\ln\frac{M_W^2}{m_\mu^2}\right)+O(L)\right\}\approx 0.02\%. \quad (17)$$

The cancellation of terms proportional to L^3 is a manifestation of the cancellation of mass singularities. The quantity in (17) is an order of magnitude smaller than the single-loop corrections of quantum electrodynamics:[11]

$$-\frac{\alpha}{2\pi}\left(\pi^2-\frac{25}{4}\right)\approx -0.4\%. \quad (18)$$

In the calculations, we allowed for the identity of the leptons in the final state. As it turns out, however, there is no contribution to the amplitude from the interference incorporating this identity in the doubly logarithmic approximation. Making use of this fact, we can extend the result derived here to five-lepton decays of the τ meson. In a doubly logarithmic approximation we have

$$\frac{d^3\Gamma(\tau\to\mu\mu\bar{\mu}\nu\bar{\nu})}{\Gamma_0\,dx_1dx_2dy}=\frac{1}{2}\,L_\mu^2\left(\frac{\alpha}{\pi}\right)^2(1+P_{12})f_1,\quad \frac{d^3\Gamma(\tau\to ee\bar{e}\nu\bar{\nu})}{\Gamma_0\,dx_1dx_2dy}=\frac{1}{2}\,L_e^2\left(\frac{\alpha}{\pi}\right)^2(1+P_{12})f_1,$$

$$\frac{d^3\Gamma(\tau\to e\bar{e}\mu\nu\bar{\nu})}{\Gamma_0\,dx_1dx_2dy}=L_e^2\left(\frac{\alpha}{\pi}\right)^2 f_1,\quad \frac{d^3\Gamma(\tau\to\mu\bar{\mu}e\nu\bar{\nu})}{\Gamma_0\,dx_1dx_2dy}=L_\mu^2\left(\frac{\alpha}{\pi}\right)^2 f_1, \tag{19}$$

where

$$L_\mu=\ln\left(\frac{m_\tau^2}{m_\mu^2}\right)\approx 5.6,\quad L_e=\ln\left(\frac{m_\tau^2}{m_e^2}\right)\approx 16.3.$$

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Translated by D. Parsons