Toward the theory of turbulent diffusion

K. V. Chukbar
Kurchatov Institute Russian Science Center, 123182 Moscow, Russia

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It is hypothesized that the relative diffusion of two particles in a turbulent velocity field with a broad spectrum (specifically, a Kolmogorov spectrum) is described by an integral equation rather than a diffusion equation. In this case the probability that the particles will quickly move far apart is a small quantity only in a power-law sense.

In this letter we wish to propose a new model-based method for describing turbulent diffusion, i.e., the spreading of a cloud of a passive impurity in a medium with intense, random velocity fluctuations which are excited over a broad spectral range. The existence of this "inertial interval" $l_{\min} < l < l_{\max}$ affects the very way in which the effect is described.[1] In this case, in contrast with ordinary diffusion, it is extremely inconvenient to follow the path traced out by one particle as it moves away from its original position. The reason is that, as long as the distance traversed by the particle remains in the inertial interval, a regular transport by the largest-scale ($l_{\max}$) velocity fluctuation is dominant. The other components of the spectrum do nothing more than add some "jitter" to the trajectory of a test particle. For this reason, it is the process of relative diffusion—the increase in the distance between two originally close particles—which is studied, both theoretically and experimentally. Accordingly, this parasitic effect is subtracted. This two-particle description of turbulent mixing is also convenient because the difference in behavior of two neighboring particles determines the shape of the cloud of passive impurity[2,3] and its fractal characteristics.[3,4] In this case the turbulence is assumed to be homogeneous and isotropic (more often in the theoretical papers than in the experimental ones).

The first result in this direction (indeed, the first quantitative result in the field of turbulence) was obtained in 1926 by Richardson.[5] Working from an analysis of experimental data, he concluded that the rate of change of the mean square distance between particles obeys a "4/3 law";

$$d\langle l^2\rangle/dt \propto (\langle l^2\rangle^{1/2})^{4/3} \tag{1}$$

(the angle brackets mean an average over many repeated experiments). It is not difficult to see that this law correlates well with the Kolmogorov–Obukhov law, according to which the velocity fluctuations in the inertial interval have a spectrum

$$v_l \propto l^{1/3}.$$

Consequently, the law in (1) corresponds to a case in which the movement of particles away from each other is dominated by fluctuations which are of the same order of magnitude as the current distance between particles:

$$d\langle l\rangle/dt = v_l \propto \langle l\rangle^{1/3}, \quad \langle l\rangle \propto t^{3/2}. \tag{2}$$

Richardson's empirical law thus has a solid theoretical footing.[1] It is usually regarded as a scientifically established fact.

Richardson, however, did not content himself with determining the important average characteristics of turbulent transport. He moved on to a study of its statistical properties. For this purpose, he introduced an analog of a two-particle distribution function: the probability density that two particles which are close together at time $t=0$ will be at the ends of a vector $\mathbf{l}$ at time t, i.e., $T(\mathbf{l},t)$ (by definition, we thus have $\int Td^3\mathbf{l}\equiv 1$). For this probability density he postulated the "kinetic equation"[5]

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial \mathbf{l}}l^{4/3}\frac{\partial T}{\partial \mathbf{l}} \tag{3}$$

(from this point on, all equations are dimensionless). The reasons why the diffusion equation was selected were apparently the simplicity of solving it, the need to achieve a spreading of T in the course of the evolution (i.e., the need to achieve a loss of information in the course of the turbulent mixing), and the familiarity of this equation. The functional dependence of the diffusion coefficient in (3) is determined in this case by the need to satisfy the 4/3 law. [Strictly speaking, the choice of diffusion coefficient in (3), like some similar operations below, satisfies the 4/3 law not only for $\langle l^2\rangle$ but also for all moments $\langle l^\alpha\rangle$ which exist.] Still, this is not a severe restriction, since the self-similar nature of T in this problem is only the flip side of the "loss of information." Richardson assumed that this quantity should be a function of l alone, and absolutely never of t. In 1952, in contrast, Batchelor reached the conclusion that this should be a functional dependence on t alone and should thus be of the form $\langle l\rangle^{4/3}\propto t^2$ (Ref. 2). In 1984, Hentshel and Procaccia[3] suggested a hybrid version: a power function in both t and l, with some latitude in the choice of powers, i.e., an extra parameter which can be adjusted to conform with experiment. We should point out that the constructive aspects of the competing arguments, in contrast with their critical aspects, suffer from a heuristic flavor and a total lack of rigor (Batchelor acknowledged this point explicitly[2]).

In this letter we would like to point out that there is no real need to restrict a kinetic description of turbulent diffusion to specifically a diffusion equation, since two of the three reasons for choosing a diffusion equation (familiarity being the exception) hold for a far broader range of integral equations. Indeed, one could make a strong argument *against* the diffusion equation: It usually applies when the "effective mean free path" of a particle is much smaller than the macroscopic length scale of the equation. From the standpoint of the problem at hand, this case corresponds to the case in which the random relative "jitter" of the particles in the turbulence field is small in comparison with the distance between particles. Actually, as was mentioned above, the situation which prevails is directly the opposite: Particles separated by a distance l are moved farther away from each other by a velocity fluctuation which is regular at this scale. This result means that the effective mean free path here is at all times comparable in magnitude to a macroscopic length scale. A process of this sort is well known in physics: radiation transport in lines in a low-density plasma.[6] The Biberman–Holstein equation which describes this transport is an integral equation of the convolution type; i.e., it is local in **k** space, not **r** space. The simple analog in this field is

$$\partial T_{\mathbf{k}}/\partial t = -|\mathbf{k}|^{2/3} T_{\mathbf{k}}, \tag{4}$$

which corresponds to

$$\frac{\partial T(\mathbf{l},t)}{\partial t} = \frac{\sqrt{3}}{4\pi^2}\Gamma(2/3)\Delta \int \frac{T(\mathbf{l}',t)}{|\mathbf{l}-\mathbf{l}'|^{5/3}} d^3\mathbf{l}'. \tag{5}$$

It is not difficult to see that solving this equation is very simple (because it is local in **k** space). Its solution has the required loss-of-information property since, as t increases, the only substantial contributions to this equation come from harmonics which are progressively closer to the zeroth (as in the usual diffusion equation). The power of 2/3 of course arises from the need to satisfy Richardson's law.

The most important difference between solution (5) and (3) is the qualitative change in behavior in the limit $l \to \infty$. The (linear) diffusion equation always has an exponentially decaying self-similar solution (with different arguments, depending on the form of the diffusion equation). In other words, the T corresponding to this process has finite moments $\langle l^\alpha \rangle$ for any positive α. The solution of (5), in contrast,

$$T(\mathbf{l},t) = \frac{1}{2\pi^2 t^{3/2}} \int_0^\infty \exp(-\kappa^{2/3}) \frac{\sin\kappa\xi}{\xi} \kappa d\kappa, \quad \xi = \frac{l}{t^{3/2}}, \tag{6}$$

has the following behavior in the limit $l \to \infty$ $(\xi \gg 1)$:

$$T \simeq \frac{\sqrt{3}}{4\pi^2}\Gamma(8/3)\frac{t}{l^{11/3}}. \tag{7}$$

In other words, it describes a process in which all moments with $\alpha \geqslant 2/3$ are infinite.

These "average" infinities of course do not mean infinite values in a real experiment. They mean that particles move away from each other extremely rapidly and in a great variety of ways. This circumstance should greatly promote a fractal structure for an impurity cloud spreading out in a turbulent medium.[3,4]

We might note that, as time elapses, the power-law "tail" in (7) leads to a linear (in time) increase in the probability that the particles have moved apart a distance greater than some fixed $l_0 \gg t^{3/2}$. This property holds for an arbitrary spectrum (not necessarily a Kolmogorov spectrum) of homogeneous and isotropic turbulence. When we take the normalization of T into account, we find directly from (5) that the right side does not depend on t in the limit $|\mathbf{l}| \to \infty$. Because the process is not local, the tail is dominated by the particles which are immediately incident from the region $\xi < 1$. In other words, the "flux at large scales" is constant (if this concept applies at all to integral equations).

In general, after one uses a nonlocal description of turbulent mixing in space, one is tempted to go through a corresponding operation with a time-varying operator on the left side of (5). However, that approach, while changing the equation, does not change its dispersive properties. Solution (6) thus remains in force. If, on the other hand, we follow Batchelor and introduce an explicit time dependence on the right side, then we of course end up with a different power in (7). As always, experiment will have the last word on the matter.

Unfortunately, I have been unable to find in the literature any data which could be used to test these data. The basic thrust of recent research in this field has been to measure the "probability density function" (see, for example, Refs. 7 and 8, which are typical examples of actual and numerical experiments). This function provides information on the statistics of fluctuations, but not on the spatial distribution of a passive impurity in the course of turbulent mixing.

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