High frequency stochastic resonance in periodically driven systems

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High frequency stochastic resonance (SR) phenomena, associated with fluctuational transitions between coexisting periodic attractors, have been investigated experimentally in an electronic model of a single-well Duffing oscillator which is bistable in a nearly resonant field of frequency ω_F. It is shown that, with increasing noise intensity, the signal-to-noise ratio (SNR) for a signal due to a weak trial force of frequency $\Omega \sim \omega_F$ at first decreases, then increases, and finally decreases again at higher noise intensities: The behavior is similar to that observed previously for conventional (low-frequency) SR in systems with static bistable potentials. The stochastic enhancement of the SNR of an additional signal at the mirror-reflected frequency $|\Omega - 2\omega_F|$ has also been observed, in accordance with theoretical predictions. Relationships with phenomena in nonlinear optics are discussed.

The first decade of research on *stochastic resonance* (SR), in which the signal due to a weak (trial) periodic force in a nonlinear system can be optimally amplified by the introduction of external noise of appropriate intensity, has concentrated almost exclusively on systems with coexisting attractors corresponding to the minima of a symmetrical, static, bistable potential. In such cases, the noise-induced amplification arises through the occurrence of fluctuational transitions between the attractors. For suitably chosen noise intensity, these transitions become nearly periodic at the frequency of the trial force, with an average amplitude that can approach the half-separation of the attractors. This is the mechanism responsible for SR phenomena studied in connection with ice ages, ring lasers, electronic circuits, passive optical systems, electron spin resonance, sensory neurons, a magnetoelastic ribbon, and a laser with saturable absorber; we shall refer to it as conventional SR.

More recently, efforts have been initiated to search for SR in other classes of systems having different kinds of attractors: It has now been identified in underdamped nonlinear oscillators that have single static attractors,[2] in a bistable, random, electrical circuit with two coexisting attractors, of which one is a limit cycle and the other is random,[3] and in a system with two coexisting stable limit cycles that have the *same period* and correspond to periodically forces vibrations of a damped nonlinear oscillator.[4] The onset of a resonant absorption that increases extremely sharply with noise intensity—a phenomenon that has much in common with SR—has been previously demonstrated theoretically[5] for the latter system. In this letter we shall show that SR of the latter kind, while similar to a conventional SR in some respects, also has a number of interesting features which distinguish it from earlier manifestations of the phenomenon and which have implications for four-wave mixing in nonlinear optics. We expect these ideas to be applicable to a large class of optically bistable, passive systems and, in particular, to optically bistable microcavities.[6]

The system which we are considering is the nearly-resonantly-driven, underdamped, single-well Duffing oscillator with additive noise,

$$\begin{aligned}&\ddot{q}+2\Gamma\dot{q}+\omega_0^2q+\gamma q^3=F\cos(\omega_Ft)+f(t),\\&\Gamma,|\delta\omega|\ll\omega_F,\quad \gamma\delta\omega>0,\quad \delta\omega=\omega_F-\omega_0,\\&\langle f(t)\rangle=0,\quad \langle f(t)f(t')\rangle=4\Gamma T\delta(t-t').\end{aligned}\tag{1}$$

This system is of topical interest because of its importance in nonlinear optics,[7] and because of its relevance to experiments on confined relativistic electrons excited by cyclotron resonant radiation.[8] If $F^2\ll\omega_0^4(\delta\omega^2+\Gamma^2)/|\gamma|$, and if the noise is weak, the comparatively small, resultant amplitude $[\ll(\omega_0^2/|\gamma|)^{1/2}]$ oscillations of $q(t)$ can conveniently be discussed in terms of the dimensionless parameters:[5]

$$\eta=\Gamma/|\delta\omega|,\quad \beta=\frac{3|\gamma|F^2}{32\omega_F^3(|\delta\omega|)^3},\quad \alpha=3|\gamma|T/8\omega_F^3\Gamma,\tag{2}$$

which characterize, respectively, the frequency detuning, the strength of the main periodic field, and the noise intensity. The bistability[9] in which we are interested arises for a restricted range of η and β: within the triangular region bounded by the solid lines in Fig. 1. Thus, as the amplitude of the periodic force is gradually increased from a small value at a fixed frequency (see the vertical line **a**–**a'**), the system moves from monostability (one small limit cycle) to bistability (two possible limit cycles of different radii), and then back again to monostability (one large limit cycle). The effect[5] of weak noise $f(t)$ involves small vibrations about the attractors and induces occasional transitions (cf. Ref. 10) between them when the system is in the bistable regime. We see that SR phenomena occur in close vicinity of the kinetic phase transition (KPT) line,[11] shown by a dashed line in Fig. 1, where the populations of the two attractors are equal.

Our principal goal is to consider the response of the system (1) to an additional weak trial force $A\cos(\Omega t+\phi)$. The combined effects of dissipation and noise result in a steady statistical distribution, and the response of the system can therefore be described in terms of linear response theory by a susceptibility. The trial force beats with the main periodic force and thus gives rise to vibrations of the system, not only at Ω but also at the combination frequencies $|\Omega\pm2n\omega_F|$ [and also at $|\Omega\pm(2n+1)\omega_F|$ in the case of nonlinearity of a general type]. We are interested in the case where the strong and trial forces are nearly resonant, i.e., the case in which ω_F and Ω lie close to ω_0. This is the case for which the response to the trial force is strongest. It is most pronounced at frequency Ω and at the nearest resonant combination, which for (1) is

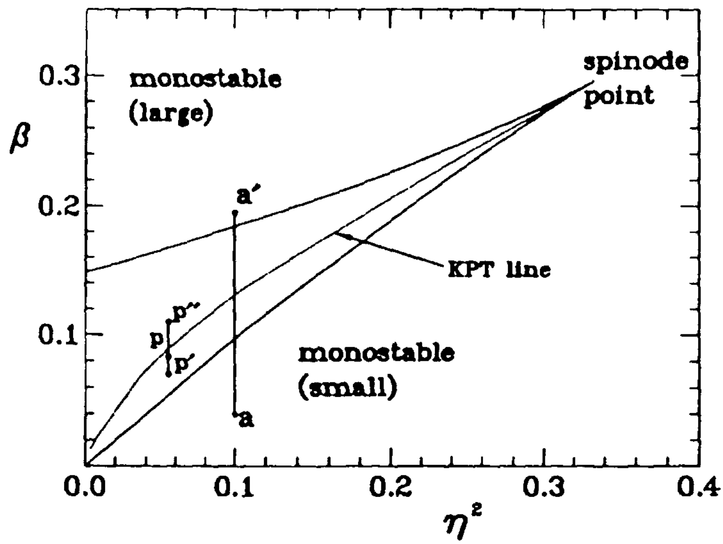

FIG. 1. Phase diagram for (1) in terms of reduced parameters (2): the cuts **a–a'**, **p'–p–p''** are discussed in the text.

$|\Omega-2\omega_F|$. The amplitudes of vibrations at these frequencies can be described respectively by the susceptibilities $\chi(\Omega)$ and $X(\Omega)$, so that the trial-force-induced modification of the coordinate q, averaged over noise, can be sought in the form

$$\delta\langle q(t)\rangle = A\mathrm{Re}\{\chi(\Omega)\exp[-i\Omega t - i\phi] + X(\Omega)\exp[i(2\omega_F-\Omega)t] - i\phi]\}. \quad (3)$$

Within the KPT range, for Ω close to ω_F, $|\mathrm{Im}\chi(\Omega)]|$ becomes large and strongly noise-dependent.[5] The rapid rise in the susceptibility with noise intensity corresponds precisely to SR since, according to (3), the squared amplitudes of the signals at frequencies Ω and $|\Omega-2\omega_F|$ (and the integrated powers of the corresponding peaks in the power spectrum) are

$$S(\Omega)=\frac{1}{4}A^2|\chi(\Omega)|^2, \quad S(|\Omega-2\omega_F|)=\frac{1}{4}A^2|X(\Omega)|^2. \quad (4)$$

An intuitive understanding of the mechanism of stochastic amplification can be gained by noting that the trial force modulates the driving force [and the coordinate $q(t)$ at the frequency $|\Omega-\omega_F|$, and that when $|\Omega-\omega_F|$ is small, the system responds almost adiabatically. In terms of the phase diagram in Fig. 1, the beat envelope then results in a slow vertical oscillation of the operating point. If the operating point is assumed to be **p**, and its range of modulation **p'–p''** is set to straddle the KPT line as shown, and the noise intensity is optimally chosen, then the system will have a tendency to make inter-attractor transitions *coherently*, once per half-cycle of the beat frequency. The net effect of the noise is therefore to increase the modulation depth of the beat envelope of the response, thereby increasing the components of the signal at the frequencies Ω and $|\Omega-2\omega_F|$.

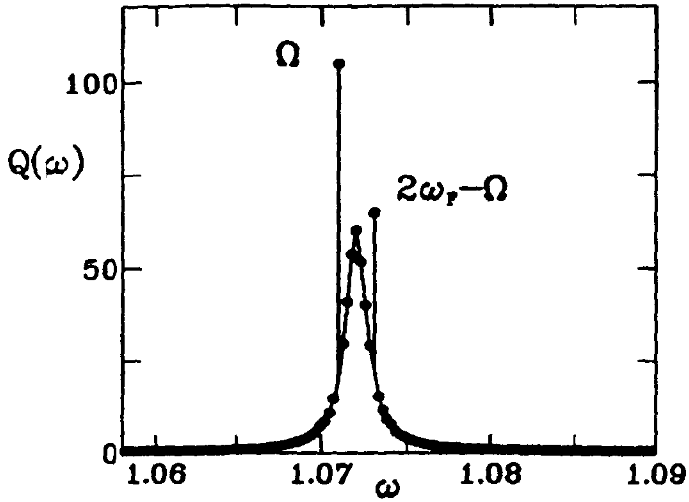

FIG. 2. Power spectral density $Q(\omega)$ measured for the electronic model, with the content of each FFT memory address shown as a separate data point on a greatly expanded abscissa; a smooth curve, which peaks at ω_F, is drawn through the background spectrum; vertical lines indicate the delta spikes resulting from the trial force.

We are investigating the response of the system (1) and the variation of the signal-to-noise ratio with α, through analog experiments on an electronic model of (1), whose relevant technical details were given in a recent conference report.[4] In terms of scaled units, the circuit parameters were set typically to $2\Gamma=0.0397$; $\omega_0=1.00$; $\gamma=0.1$; $\omega_F=1.07200$; $\Omega=1.07097$; and to seek SR near the KPT, $F=0.068$ and the amplitude of the trial force $A=0.006$. A spectral density of fluctuations of the coordinate $q(t)$ [about $\langle q(t)\rangle$] for $A=0$ recorded with the above parameter values for $\alpha=0.061$ and 16 384 samples in each realization is shown in Fig. 2. The smooth background is the supernarrow peak of Ref. 11, here broadened by noise (although its width remains very much smaller than 2Γ); the delta function spikes, indicated by raised points[12] of the discrete spectrum, are clearly visible, not only at the trial force frequency (Ω), but also as predicted at the mirror-reflected frequency ($2\omega_F-\Omega$).

The signal-to-noise ratios R, determined in the usual way[1] from measurements of the delta spikes and from the smooth background, are plotted (data points) as functions of the noise intensity α in Fig. 3 for $\beta=0.814$ and $\eta=0.236$. It is immediately evident that there is a range of α within which R *increases* with α. It is also apparent that for the main and the mirror-reflected signals the shape of the $R(\alpha)$ curve in Fig. 3 is remarkably similar to that observed earlier[1] in the case of conventional SR. In other words, R initially decreases with α as a result of the increase in its denominator. With a further increase of α, the inter-attractor transitions come into play and become phase-coherent with the trial force to a high probability, giving rise to an increase in R through the stochastic amplification mechanism discussed above. Finally, for still larger α, the value of R decreases again, partly because of the continuing increase in

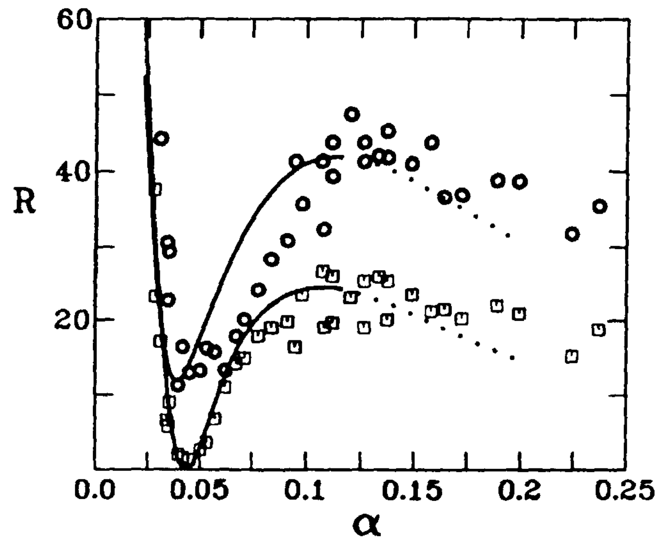

FIG. 3. The signal-to-noise ratio R of the response of the system (1) to a weak trial force at a frequency Ω, as a function of noise intensity α, in experiment and theory: at the trial frequency Ω (circles and associated theoretical curve); and at the "mirror-reflected" frequency $(2\omega_F-\Omega)$ (squares). For noise intensities near those of the maxima in $R(\alpha)$, the asymptotic theory is only qualitative (shown by dotted curves).

its denominator and partly because of the frequent occurrence of the transitions within individual periods of the trial force, with a corresponding reduction in the proportion of the phase-coherent jumps which are responsible for the amplification.

A quantitative theory of this phenomenon can be easily constructed by extending[13] the theory of Ref. 5. This would lead to contributions to the susceptibilities from interattractor transitions of the form

$$\chi_{tr}(\Omega)=\frac{w_1w_2}{2\omega_F(\omega_F-\omega_0)}(u_1^*-u_2^*)\frac{\mu_1-\mu_2}{\alpha}\left[1-\frac{i(\Omega-\omega_F)}{W_{12}+W_{21}}\right]^{-1},$$
$$X_{tr}(\Omega)=\frac{u_1-u_2}{u_1^*-u_2^*}\chi_{tr}(\Omega),\quad \mu_j=\sqrt{\beta}\left(\frac{\partial R_j}{\partial\beta}\right), \tag{5}$$

where w_1 and w_2 are the populations of the attractors 1 and 2, and W_{12} and W_{21} are the probabilities for the transitions between them, which are of the activation type, $W_{ij}\propto\exp(-R_i/\alpha)$. (The u_i and u_i^*, which define the positions of the attractors in the rotating coordinate frame, can be regarded as constants for given η and β.) It is evident that the contributions (5) come into play if, and only if, the system is in the KPT range, in which the populations of the unperturbed attractors are comparable: otherwise, the factor $w_1w_2\propto\exp(|R_1-R_2|/\alpha)$ is exponentially small. In the KPT range, however, the susceptibilities are large because they are proportional to the large factor $|\mu_1-\mu_2|/\alpha$; the rapid increase of W_{ij} with the noise intensity then ensures that there will be a range of α in which both susceptibilities increase very rapidly with α, consistent with the experiments. The complete theory,[13] including the effect of intra-attractor vibrations, leads to the curves in Fig. 3. Given the large systematic errors inherent in the measurements—arising, e.g., from $\delta\omega$ (1), a small difference between large quantities which, in β (2), is then raised to the third power—the agreement between theory and experiment is considered excellent.

In conclusion, we would emphasize (first) that, in contrast with the earlier forms of bistable SR,[1,3] stochastic amplification occurs here *not* at the relatively low frequency of the quasi-periodic inter-attractor hopping, but rather at Ω close to the much higher (tunable) frequency, ω_F, of the main periodic driving force. To emphasize the distinction, it seems appropriate to call the new form of SR as high frequency stochastic resonance (HFSR). Secondly, we draw attention to the relationship of HFSR to four-wave mixing in nonlinear optics.[14] In effect, our results correspond to noise-enhanced amplification of the signal wave and noise-enhanced generation of the idler wave. The mechanisms are *resonant* and, although they have the appearance of four-wave mixing, they actually correspond to multiple-wave processes: In terms of quantum optics, the oscillator absorbs and re-emits many quanta of the strong field. The very high amplification/generation coefficients arise partly from their resonant character and partly from the fact that the signal sizes correspond, not to the amplitudes of vibrations about the attractors but, as is usually the case in bistable SR, to the "distance" between the attractors (to their coordinate separation for conventional SR, and approximately to the difference in amplitudes in the present case).

Finally, our prediction and demonstration of HFSR for periodic attractors, and its similarity (Fig. 3) to conventional SR, leads to a broader and more general perception of the physical nature of bistable SR. Like the onset of supernarrow peaks in the power spectra, conventional SR[1] and HFSR are both critical phenomena which arise in the range of the KPT. While HFSR is to be anticipated for coexisting, stable, limit cycles with equal periods, low-frequency SR is a more robust effect. It arises not only in systems with fluctuating simple double-well potentials but also in systems where one (or both) attractors are random.[3] We can infer from Ref. 5 that, in general, low frequency SR can be anticipated for periodic attractors [although not for (1), where the centers of forced vibrations are independent of amplitude]. Since noise gives rise to fluctuational hopping between any type(s) of attractors, it seems reasonable to conclude that SR is actually a quite general phenomenon characteristic of *all* systems with coexisting attractors, regardless to the nature of those attractors, provided that the system lies in the KPT range.

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