Solution of self-duality equation in quantum-group gauge theory

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A gauge theory for the quantum group $SU_q(2)\times U(1)$ on the quantum Euclidean space is discussed. This theory contains three physical gauge fields and one $U(1)$ gauge field with a zero field strength. We construct the quantum-group self-duality equation (QGSDE) in terms of differential forms and with the help of the field-strength decomposition. A deformed analog of the BPST instanton solution is obtained. The possibility of a harmonic (twistor) interpretation of QGSDE in terms of quantum harmonics is considered. © *1995 American Institute of Physics.*

An attractive idea of quantum deformations for the gauge theories has been considered in the framework of different approaches.[1-3] Formally one can discuss independent deformations of basic spaces and gauge groups and possible correlations between these deformations. Let us consider a gauge theory with identical one-parameter deformations of the 4-dimensional Euclidean space and the gauge group $SU(2)$. Since a consistent formulation of the gauge theory for the semi-simple quantum group $SU_q(N)$ is unknown to us, we shall deal with the quantum group $U_q(2)=SU_q(2)\times U(1)$. It will be shown that the $U(1)$ gauge field can be treated as a field with a zero field strength.

Consider the standard relations between the elements T^i_k of the quantum $U_q(2)$ matrix[4]

$$RTT'=TT'R\Leftrightarrow R^{ik}_{lm}T^l_jT^m_n=T^i_lT^k_mR^{lm}_{jn}, \tag{1}$$

where I is a unit matrix, R is a constant symmetric matrix with the components $R^{ik}_{lm}(q)$ $(i,\ k,\ l,\ m=1,\ 2)$, and q is a real deformation parameter. We use the R-matrix method in the condensed notation of Ref. 3 (see also Refs. 4, and 5). A translation of matrix formulas to the usual index notation can be done with the help of the following substitutions:

$$R\Rightarrow R^{ik}_{lm},\quad T\Rightarrow(T\otimes I)^{lm}_{nj}=T^l_n\delta^m_j,\quad T'\Rightarrow(I\otimes T)^{nj}_{ps}=\delta^n_pT^j_s. \tag{2}$$

The parameters $q(ik)$ $[q(12)=q,\ q(21)=q^{-1},\ q(11)=q(22)=1]$ define a q deformation of the ε symbol $\varepsilon_{ik}(q)\ =\ \sqrt{q(ik)}\varepsilon_{ik}$, where ε_{ik} is an ordinary antisymmetric symbol.

The R matrix can be written in terms of the projection operators $P^{(\pm)}$: $R=qP^{(+)}-q^{-1}P^{(-)}$, $P^{(+)}+P^{(-)}=I$. The operator $P^{(-)}$ for $U_q(2)$ is proportional to the product of two $\varepsilon(q)$ symbols:

$$[P^{(-)}]^{ik}_{lm}=-\frac{q}{1+q^2}\varepsilon^{ki}(q)\varepsilon_{ml}(q), \quad \varepsilon^{ki}(q)\varepsilon_{il}(q)=\delta^k_l. \tag{3}$$

Here we also write the basic identity for the $\varepsilon(q)$ symbols with the upper indices.

We shall use the covariant relation for the quantum determinant $D(T)$ of the $U_q(2)$ matrix

$$\varepsilon_{ml}(q)T^l_iT^m_k=\varepsilon_{ki}(q)D(T). \tag{4}$$

A covariant expression for the inverse quantum matrix $S(T)=T^{-1}$ can be obtained from this equation.

The $SU_q(2)$ metric $\mathscr{D}(q)$ determines the matrix product of the transposed q matrices[4]

$$T^l_i\mathscr{D}^m_l(q)S(T)^k_m=\mathscr{D}^k_i(q)=-\varepsilon_{mi}(q)\varepsilon^{mk}(q). \tag{5}$$

The unitarity condition for the matrix T can be formulated with the help of the involution[4] $\overline{T^i_k} = S^k_i$.

Let us consider the bicovariant differential calculus on the $U_q(2)$ group:[5–7]

$$T\ dT'=R\ dT\ T'\ R, \quad D(T)\ dT=q^2dT\ D(T). \tag{6}$$

Note that the condition $D(T)=1$ is inconsistent in the framework of this calculus. Consider the relations for the right-invariant differential forms $\omega=dTS$ (Ref. 3)

$$\omega\ R\omega+R\omega\ R\omega\ R=0, \quad T\omega'=R\omega\ RT. \tag{7}$$

The quantum tract ξ of the form ω plays an important role in this calculus:

$$\xi(T)=\mathscr{D}^k_i(q)\omega^i_k(T)\neq 0, \quad \xi^2=0, \quad d\xi=0, \tag{8}$$

$$dT=\omega T=(q^3-q)^{-1}[T,\xi], \quad qdD(T)=\xi D(T), \quad d\omega=\omega^2=(q-q^3)^{-1}\{\xi,\omega\}. \tag{9}$$

Note that the basic relations of the bicovariant calculus on $GL_q(2)$ and $U_q(2)$ were analyzed in detail in Refs. 5–7. We call this calculus the BC calculus.

The BC calculus makes the basis for consistent formulation of the quantum-group gauge theory in the framework of noncommutative algebra of differential complexes.[2,3] Consider formally the quantum group gauge matrix $T^a_b(x)$ defined on some basis space. Let us assume that Eqs. (4)–(6) are satisfied locally at each "point" x. We can then construct the $U_q(2)$ connection 1-form $A^a_b(x)$ which satisfies a simple commutation relation

$$(A\ R\ A+R\ A\ R\ A\ R)^{ab}_{cd}=A^a_eR^{eb}_{gd}A^g_c+R^{ab}_{ef}A^e_gR^{gf}_{hn}A^h_mR^{mn}_{cd}=0. \tag{10}$$

These relations generalize the anticommutativity conditions for the components of the classical connection form. Note that the general relation for A contains a nontrivial right-hand side.[3]

Coaction of the gauge quantum group $U_q(2)$ has the standard form

$$A \to T(x)AS(T) + dT(x)S(T) = TAS + \omega(T),$$
$$\alpha = \mathrm{Tr}_q A \to \alpha + \xi(T). \tag{11}$$

The restriction $\alpha=0$ is inconsistent with (10), but we can use the gauge-covariant relations $\alpha^2=0$ and $\mathrm{Tr}_q A^2=0$.

It should be noted that we can choose the zero field strength condition $d\alpha=0$ for the $U(1)$ gauge field[2]. This constraint is gauge invariant and consistent with (10). The deformed pure gauge field α can be decoupled from the set of physical fields in the limit $q=1$. We shall consider further the $U_q(2)$ gauge theory with three "physical" gauge fields and one "zero-mode" $U(1)$ field. The curvature 2-form $F=dA-A^2$ is q-traceless for this model.

Quantum deformations of Mikowski and Euclidean four-dimensional (4D) spaces have been considered in Refs. 8 and 9. We shall treat the coordinates x^i_α of q-deformed Euclidean space $E_q(4)$ as generators of a noncommutative algebra $(Rxx'=xx'R)$ covariant under the coaction of the quantum group $G_q(4)=SU^L_q(2)\times SU^R_q(2)$. The q-deformed central Euclidean interval τ can be constructed by analogy with the quantum determinant

$$\tau = x^1_1 x^2_2 - q x^1_2 x^2_1 = -\frac{q}{1+q^2}\varepsilon^{\beta\alpha}(q)\varepsilon_{ki}(q)x^i_\alpha x^k_\beta. \tag{12}$$

We do not consider the quantum group structure on $E_q(4)$. It is convenient to use the following $E_q(4)$ involution

$$\overline{x^i_\alpha} = \varepsilon_{ik}(q)x^k_\beta \varepsilon^{\beta\alpha}(q) = \tau S^\alpha_i(x), \tag{13}$$

$$\overline{\tau} = \tau, \quad \overline{\overline{x^i_\alpha}} = x^i_\alpha, \tag{14}$$

where $S(x)$ is an inverse matrix for the matrix x.

We shall use an analog of the bicovariant $U_q(2)$ calculus for studying differential complexes on $E_q(4)$. The commutation relations between matrices x and dx can be obtained from Eqs. (6)–(9) by formal substitution $T\to x$. We can obtain, for example, the relations

$$x^i_\alpha dx^k_\beta = R^{ik}_{lm} dx^l_\gamma x^m_\rho R^{\gamma\rho}_{\alpha\beta},$$
$$P^{(+)}dxdx'P^{(+)} = 0 = P^{(-)}dxdx'P^{(-)}. \tag{15}$$

The basic decomposition of the 2-forms on $E_q(4)$ is

$$dx^i_\alpha dx^k_\beta = [P^{(-)}dxdx' + dxdx'P^{(-)}]^{ik}_{\alpha\beta} = \varepsilon^{ki}(q)d^2x_{\alpha\beta} + \varepsilon_{\beta\alpha}(q)d^2x^{ik}, \tag{16}$$

where Eq. (3) for $P^{(-)}$ is used. By analogy with the classical case, we can treat two terms of this decomposition as self-dual and anti-self-dual 2-forms under the action of a duality operator $*$.

Consider the right-invariant 1-forms on $E_q(4)$

$$\omega^i_k(x) = [dxS(x)]^i_k = dx^i_\alpha S^\alpha_k, \quad dx = \omega x, \tag{17}$$

where $S(x)$ is the inverse matrix for x defined by Eq. (13). It is convenient to rewrite the decomposition of 2-forms in terms of the right-invariant self-dual and anti-self-dual forms

$$dxdx' = \omega x \omega' x' = \omega R \omega R x x', \tag{18}$$

$$P^{(-)} dxdx' = q P^{(-)} \omega R \omega x x' P^{(+)} = P^{(-)} \Omega_S P^{(+)} x x', \tag{19}$$

$$dxdx' P^{(-)} = \Omega_A P^{(-)} x x', \tag{20}$$

$$\Omega_S = *\Omega_S = q^4 \omega^2 + q \omega \xi, \tag{21}$$

$$\Omega_A = -(*\Omega_A) = q^{-1} \omega \xi - \omega^2. \tag{22}$$

Here the commutation relations of BC-calculus on $E_q(4)$ and the properties of the $P^{\pm}$ operators were used. It should be stressed that the condensed notation simplifies significantly these calculations.

Let us introduce the simple ansatz for quantum $U_q(2)$ anti-self-dual gauge fields

$$\begin{aligned} A^a_b &= dx^i_\alpha A^{\alpha a}_{ib}(x) = \omega^a_b(x) f(\tau), \\ A^{\alpha a}_{ib}(x) &= \delta^a_i S^\alpha_b(x) f(\tau), \end{aligned} \tag{23}$$

where $f(\tau)$ is a function of the q-interval (12). Note that this ansatz is a partial case of the more general construction of the differential complex on $GL_q(2)$ (Refs. 2 and 3). Addition of the term $\xi(x)g(\tau)$ gives a relation for the connection A that is more complicated than (10).

Consider the q-traceless curvature form for the connection (23) which can be calculated in the framework of the BC-calculus on $E_q(4)$:

$$F = \omega^2 f(\tau)[1 - f(q^2\tau)] + (q^3 - q)^{-1} \omega \xi [f(\tau) - f(q^2 \tau)]. \tag{24}$$

The appearance of the finite translation $f(q^2\tau)$ is a general feature of the calculus on the quantum space.

The anti-self-duality equation for our ansatz is equivalent to the nonlinear finite-difference equation

$$*F = -F \Rightarrow F \sim \Omega_A f(\tau)[1 - f(q^2\tau)], \tag{25}$$

$$f(\tau) - f(q^2\tau) = (1 - q^2) f(\tau)[1 - f(q^2\tau)], \tag{26}$$

where Ω_A is the anti-self-dual 2-form [Eq. (22)].

This equation has a simple solution which is analogous to the classical BPST solution:

$$f(\tau) = \frac{\tau}{a + \tau}, \tag{27}$$

where a is an arbitrary "constant" that can be treated as a central periodical function: $a(\tau) = a(q^2\tau)$. Note that our solution for the connection A contains the parameter q only through the definitions of $\omega(x)$ and τ. However, the corresponding curvature has a more explicit q-dependence (24).

It is easy to obtain the 5-parameter solution via the substitution[3)] $x^i_\alpha \to \hat{x}^i_\alpha = x^i_\alpha + c^i_\alpha$ in Eqs. (23) and (27). The note that our anti-self-dual solution is a function on the braided algebra with the noncommuting generators x, dx, a, and c.

$$R\hat{x}\hat{x}' = \hat{x}\hat{x}'R, \quad Rcc' = cc'R, \quad cx' = Rxc'R, \tag{28}$$

$$cdx' = Rdxc'R, \quad [\hat{x}, \tau(\hat{x})] = 0,$$

$$d\hat{x} = dx, \quad dc = 0, \tag{29}$$

$$\tau(\hat{x})dx = q^2 dx\,\tau(\hat{x}).$$

The QGSD equation can be written in terms of the field strength:

$$F = dx^i_\alpha dx^k_\beta F^{\beta\alpha}_{ki}(x), \quad F^{\beta\alpha}_{ki} = \varepsilon_{ki}(q)F^{\beta\alpha}. \tag{30}$$

We introduce the additional noncommutative harmonic twistor variables u^i_+ which satisfy the relations $\varepsilon_{ki}(q)u^i_+ u^k_+ = 0$. We can obtain the integrability condition by multiplying the QGSD equation by the product $u^i_+ u^k_+$. The analogous integrability conditions are the basis of the harmonic (twistor) approach to the classical self-duality equation.[10,11] We considered the deformed harmonic formalism of QGSDE in Ref. 12.

It seems very interesting to study the reductions of QGSDE to lower dimensions and to search for a more general deformation scheme for the self-duality equation.

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[2)]This condition is consistent for the case of $GL_q(N)$ group.

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