Relationship between the coefficient functions for annihilation and deep inelastic processes

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The one-loop nature of the axial anomaly, which is seen when the axial current is normalized in the appropriate way, is shown to be the reason for the cancellation of corrections of the type $C_F^N \bar{\alpha}_s^N$ $(N \geq 1)$ in the Crewther relation for the coefficient functions for annihilation and deep inelastic processes. © *1995 American Institute of Physics.*

The status of the Crewther relation[2] was analyzed within the framework of QCD in Ref. 1. Recent results from multiloop calculations of coefficient functions for the sum rules for deep inelastic scattering and for e^+e^- annihilation into hadrons were taken into account. Some extremely interesting properties of this relation were pointed out in Ref. 1. In particular, it was shown that corrections of the type $C_F\bar{\alpha}_s$, $C_F^2\bar{\alpha}_s^2$, and $C_F^3\bar{\alpha}_s^3$ cancel out in the product of a coefficient function in the Bjorken sum rule for polarized deep inelastic lepton–hadron scattering and the Adler function for the two-loop correlation function for electromagnetic currents. It was also shown that corrections in second- and third-order perturbation theory, which do not cancel out, are proportional to the two-loop β function of QCD. The relation derived in Ref. 1 is

$$C_{Bj}(\bar{a}_s)C_R(\bar{a}_s)=1+\frac{\beta^{(2)}(\bar{a}_s)}{\bar{a}_s}[K_1C_F\bar{a}_s+(K_2N_F+K_AC_A+K_FC_F)C_F\bar{a}_s^2]+O(\bar{a}_s^4). \tag{1}$$

Here $\bar{\alpha}_s=\bar{\alpha}_s(\mu^2=Q^2)/4\pi$; N_F is the number of flavors; C_A and C_F are Casimir operators (in the QCD case we have $C_A=3$ and $C_F=4/3$); and $\beta^{(2)}\times(\bar{a}_s)=\beta_1\bar{a}_s^3+\beta_2\bar{a}_s^3+O(\bar{a}_s^4)$ is the two-loop approximation for the QCD β function, which does not contain terms proportional to $C_F^{N-1}\bar{\alpha}_s^N$ $(N\geq 2)$. The coefficients in Eq. (1) are defined by

$$K_1=\left(-\frac{21}{2}+12\zeta(3)\right);\quad K_2=\left(+\frac{326}{6}-\frac{304}{6}\zeta(3)\right);\quad K_A=\left(-\frac{629}{2}+\frac{884}{3}\zeta(3)\right);$$

$$K_F=\left(+\frac{397}{6}+136\zeta(3)-240\zeta(5)\right).$$

In the Bjorken sum rule for polarized deep inelastic lepton–hadron scattering, the coefficient function C_{Bj} is found from an operator expansion of the type

$$i\int TV_\alpha(x)V_\beta(0)e^{ipx}dx|_{|p^2|\to\infty}\simeq C_{Bj}(\bar{a}_s)\,\frac{\epsilon_{\alpha\beta\rho\lambda}p^\rho}{p^2}\,\frac{1}{12}\,A^{(3)\lambda}(0)+\cdots.$$

Here V_α is the electromagnetic current, and $A^{(3)\lambda}$ is the third component of the axial isotriplet (an interpolating current for a π meson). Explicit expressions for this coefficient function have been found in the two-loop[3] and three-loop[4] perturbation-theory approximations. In leading order, this function is $C_{Bj}(\bar{a}_s)=1-3C_F\bar{a}_s+O(\bar{a}_s^2)$. The quantity C_R in (1) is related to the coefficient function for the cross section for e^+e^- annihilation into hadrons. It is also known in the two-loop[5] and three-loop[6] approximations; in the leading perturbation-theory order it is

$$C_R(\bar{a}_s)=D(\bar{a}_s)/N_c=1+3C_F\bar{a}_s+O(\bar{a}_s^2),$$

where the Adler function $D(\bar{a}_s)$ is given by

$$D(\bar{a}_s)=-12\pi^2q^2\,\frac{d}{dq^2}\,\Pi(q^2),$$

$$i\int\langle 0|TA_\alpha^{(3)}(x)A_\beta^{(3)}(0)|0\rangle e^{iqx}dx=(g_{\alpha\beta}q^2-q_\alpha q_\beta)\Pi(q^2).$$

Our purposes in the present study are to clarify the reasons for the cancellation of corrections of the type $C_F\bar{a}_s$, $C_F^2\bar{a}_s^2$, and $C_F^3\bar{a}_s^3$ in the Crewther relation and to generalize this relation to higher-order perturbation theories. As we show below, the observed cancellation stems from the particular structure of the anomalous triangle and the Adler–Bardeen theorem.[7]

We consider a three-point correlation function of the type

$$\begin{aligned}T_{\mu\alpha\beta}(p,q)&=\int\langle 0|TA_\mu^{(3)}(y)V_\alpha(x)V_\beta(0)|0\rangle e^{ipx+iqy}dxdy\\&=\zeta_1(q^2,p^2)\epsilon_{\mu\alpha\beta\tau}p^\tau\\&\quad+\zeta_2(q^2,p^2)(q_\alpha\epsilon_{\mu\beta\rho\tau}p^\rho q^\tau-q_\beta\epsilon_{\mu\alpha\rho\tau}p^\rho q^\tau)\\&\quad+\zeta_3(q^2,p^2)(p_\alpha\epsilon_{\mu\beta\rho\tau}p^\rho q^\tau+p_\beta\epsilon_{\mu\alpha\rho\tau}p^\rho q^\tau),\end{aligned}\tag{2}$$

where we are using an expansion in a basis of three independent tensor structures under the kinematic condition $pq=0$ (see Ref. 8 for a detailed discussion). Following Ref. 2, we consider an operator expansion for this correlation function in the limit $|p^2|\to\infty$. Using the relations between various tensor structures,[8] we can easily show that we have

$$T_{\mu\alpha\beta}(q,p)\to\frac{1}{12}\,\frac{1}{p^2}\,C_{Bj}(\bar{a}_s)\Pi(q^2)(q_\alpha\epsilon_{\mu\beta\rho\tau}p^\rho q^\tau-q_\beta\epsilon_{\mu\alpha\rho\tau}p^\rho q^\tau),$$

and thus

$$\zeta_2(q^2,p^2)|_{|p^2|\to\infty}\to\frac{1}{12}\,\frac{1}{p^2}\,C_{Bj}(\bar{a}_s)\Pi(q^2).\tag{3}$$

On the other hand, the requirement of gauge invariance leads to a Ward identity for this Green's function. In our case, this identity is[8]

$$-\zeta_1(q^2,p^2)=q^2\zeta_2(q^2,p^2)+p^2\zeta_3(q^2,p^2).$$

Differentiating the relation found with respect to q^2, and noting that the function ζ_1 is simply an unrenormalizable number according to the Adler–Bardeen theorem,[7] we find an equation for the two other functions:

$$q^2\frac{d}{dq^2}\zeta_2(q^2,p^2)=-p^2\frac{d}{dq^2}\zeta_3(q^2,p^2)-\zeta_2(q^2,p^2). \tag{4}$$

Actually, the one-loop nature of the anomaly is extremely conditional. In terms of an operator relation, this one-loop nature is reached when the normalizations of the axial and vector currents are reconciled by means of the relation $(\Lambda_\mu^5)^{\text{Ren}}=\gamma_5(\Lambda_\mu)^{\text{Ren}}$, where $(\Lambda_\mu^5)^{\text{Ren}}$ and $(\Lambda_\mu)^{\text{Ren}}$ are the renormalized axial and vector vertex functions. However, this condition does not yet guarantee that there are no corrections in terms of the Green's functions (in the case at hand, there are no corrections to the quantity ζ_1). As was shown in Ref. 9, contributions containing diagrams of the type corresponding to a scattering of light by light lead to corrections to the anomaly in terms of the Green's functions in second-order perturbation theory. In our case, in which the axial current in three-point correlation function (2) is a nonsinglet current in terms of flavors, diagrams of this type renormalize the quantity ζ_1 in second order in the fine-structure constant, but not in the order $\bar{a}_s^2$. Accordingly, ignoring higher electromagnetic corrections, we can postulate that ζ_1 is of a one-loop nature. At the same time, according to relation (3) we have the following relation in the limit $|p^2|\to\infty$:

$$q^2\frac{d}{dq^2}\zeta_2(q^2,p^2)\to-\frac{N_c}{(12\pi)^2}\frac{1}{p^2}C_{Bj}(\bar{a}_s)C_R(\bar{a}_s). \tag{5}$$

We can now write expansions for ζ_2 and ζ_3 as power series in q^2/p^2:

$$\zeta_s(q^2,p^2)\to\frac{1}{p^2}\sum_{k=0}^{\infty}\left(\frac{q^2}{p^2}\right)^k\zeta_2^k,\quad \zeta_3(q^2,p^2)\to\frac{1}{p^2}\sum_{n=0}^{\infty}\left(\frac{q^2}{p^2}\right)^n\zeta_3^n,$$

where ζ_2^k and ζ_3^k are dimensionless coefficients. Substituting these expressions into (4), we find

$$q^2\frac{d}{dq^2}\zeta_2(q^2,p^2)\to-\frac{1}{p^2}\sum_{k=0}^{\infty}[(k+1)\zeta_3^{k+1}=\zeta_2^k]\left(\frac{q^2}{p^2}\right)^k. \tag{6}$$

Comparing (6) with relation (3), we find the following expression for the product of C_{Bj} and C_R:

$$\frac{N_c}{(12\pi)^2}C_{Bj}(\bar{a}_s)C_R(\bar{a}_s)=\zeta_3^1+\zeta_2^0. \tag{7}$$

In leading perturbation-theory order we have $\zeta_3^1+\zeta_2^0=N_c/(12\pi)^2$. We have thus verified that the one-loop or multiloop behavior of the product $C_{Bj}C_R$ stems from a nonrenormalizability or a renormalizability, respectively, of invariant functions present in the expression for the anomalous three-point correlation function. On the other hand, it was shown in Ref. 10 that, if conformal invariance is present in the theory, then the general expression for the three-point Green's function $T_{\mu\alpha\beta}$ is of a form determined completely by the one-loop value of this function, $\Delta_{\mu\alpha\beta}$:

$$T_{\mu\alpha\beta}(p,q)=K(\bar{a}_s)\Delta_{\mu\alpha\beta}(p,q).$$

Here $K(\bar{a}_s)$ is a quantity which is undetermined in the approach of Ref. 10. In other words, in a conformally invariant theory we have, according to Ref. 10,

$$\zeta_1^{exact}=K(\bar{a}_s)\zeta_1^{one\ loop},\quad \zeta_2^{exact}=K(\bar{a}_s)\zeta_2^{one\ loop},\quad \zeta_3^{exact}=K(\bar{a}_s)\zeta_3^{one\ loop}. \tag{8}$$

However, we know that a renormalization procedure disrupts the initial conformal invariance of massless QCD and gives rise to an anomaly in the trace of the energy–momentum tensor.[2,11] The explicit expression for the anomaly of the energy–momentum tensor[11] shows that the factor $\beta(\bar{a}_s)/(\bar{a}_s)$ is a measure of the breaking of conformal invariance within the perturbation-theory framework. We can thus rewrite Eqs. (8) in QCD as follows:

$$\zeta_1^{exact}=K(\bar{a}_s)\zeta_1^{one\ loop},\quad \zeta_2^{exact}=\left[K(\bar{a}_s)+\frac{\beta(\bar{a}_s)}{\bar{a}_s}\,v_2(p^2,q^2,\bar{a}_s)\right]\zeta_2^{one\ loop},$$

$$\zeta_3^{exact}=\left[K(\bar{a}_s)+\frac{\beta(\bar{a}_s)}{\bar{a}_s}\,v_3(p^2,q^2,\bar{a}_s)\right]\zeta_4^{one\ loop},$$

where v_2 and v_3 are dimensionless functions which satisfy Ward identity (4). Now recalling that we have $\zeta_1^{\text{exact}}=\zeta_1^{\text{one loop}}$ according to the Adler–Bardeen theorem, we find $K(\bar{a}_s)=1$. The invariant functions ζ_2 and ζ_3 are thus renormalized by factors containing only the factor $\beta(\bar{a}_s)/\bar{a}_s$ in higher-order perturbation theories. This fact in turn means that the product $C_{Bj}C_R$ takes the following form in all perturbation-theory orders:

$$C_{Bj}(\bar{a}_s)C_R(\bar{a}_s)=1+\frac{\beta(\bar{a}_s)}{\bar{a}_s}\,r(\bar{a}_s),$$

where $r(\bar{a}_s)$ is a polynomial in powers of $\bar{a}_s$, which is not fixed in our approach.

In summary, we have analyzed the reasons for the cancellation of corrections of the type $C_F^N\bar{a}_s^N$ $(N\geqslant 1)$ in the product of a coefficient function from the Bjorken sum rule for polarized deep inelastic lepton–hadron scattering and the Adler function for the two-point correlation function of electromagnetic currents (the Crewther relation). We have shown that these cancellations stem directly from the nonrenormalizability of the axial anomaly when the normalization of the axial current is chosen appropriately (the Adler–Bardeen theorem). We have also shown that all the corrections which do not cancel out are proportional to the factor $\beta(\bar{a}_s)/\bar{a}_s$, which is turn a measure of the breaking of conformal invariance in QCD.

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