Relationship between the Aharonov–Bohm and Aharonov–Casher effects for various particle spins

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(Submitted 10 February 1995)
Pis'ma Zh. Éksp. Teor. Fiz. **61**, No. 6, 444–448 (25 March 1995)

The relationship between the Aharonov–Bohm and Aharonov–Casher topological effects is analyzed as a function of the spin of the moving particle and the orientation of this spin. A duality relationship between the wave functions prevails only if there is no spin precession, i.e., in the case of a definite projection, maximum in absolute value, of the spin onto the normal to the plane of motion. A generalization for particles which have both a charge and an anomalous magnetic moment is discussed. Some experiments with atomic beams are proposed.

1. The topological properties of physical systems are attracting much interest among physicists and mathematicians. The best-known example of these properties is the Aharonov–Bohm effect,[1] which is manifested in the motion of a quantum-mechanical charged particle in the presence of an infinitely thin solenoid. Years after this effect was discovered, Aharonov and Casher[2] presented some arguments for the existence of yet another configuration, which would be a dual of the Aharonov–Bohm configuration. This is the motion of a neutral point magnetic dipole in the field of a uniformly charged filament. The approximations used in Ref. 2 were examined in detail in Refs. 3–5, and they came under criticism. Recently, however, Hagen[6] has shown that for a spin-1/2 particle the Aharonov–Bohm (AB) and Aharonov–Casher (AC) configurations lead to exact equivalence of the amplitudes, although the AC cross section has a polarization dependence different from that of the corresponding AB cross section.

In the present letter we pursue Hagen's approach in two directions. Reproducing Hagen's results for spin 1/2 within the framework of the ordinary formalism, without using the explicit expressions for the Dirac matrices, we consider particles with a larger spin. We then move on to the duality problem for a particle which has both a charge and an anomalous magnetic moment.

2. We consider a Dirac particle in an AB configuration, i.e., in the field of a straight thin solenoid oriented along the third axis (the z axis). If the wave function does not depend on z, the Dirac equation becomes

$$[i\gamma_0\partial_0-\gamma_k(i\nabla_k-eA_k)-m]\Psi=0, \tag{1}$$

where k takes on the values 1 and 2. This equation has an integral of motion, which can be described by the operator $(1/2)\gamma^3\gamma^5$. The physical meaning of this integral is obvious: It is equal to the operator representing the third projection of the spin pseudovector

$$\xi^{\mu}=\frac{1}{2}\gamma^{\mu}\gamma^{5}. \quad (2)$$

We adopt eigenstates Ψ_s, for which we have

$$\gamma^3\gamma^5\Psi_s=s\Psi_s, \quad s=\pm 1. \quad (3)$$

For them we have

$$\gamma_1\Psi_s=\frac{1}{s}\gamma_1\gamma^3\gamma^5\Psi_s=\frac{1}{s}\sigma_{02}\Psi_s,$$

$$\gamma_2\Psi_s=-\frac{1}{s}\sigma_{01}\Psi_s, \quad \sigma_{\mu\nu}=\frac{i}{2}[\gamma_\mu,\gamma_\nu]. \quad (4)$$

As a result, Eq. (1) takes the form of a Dirac equation for a neutral particle with an anomalous magnetic moment,

$$\left(i\gamma^{\kappa}\partial_{\kappa}-m-\frac{\mu}{2}F'^{\alpha\beta}\sigma_{\alpha\beta}\right)\Psi_s=0 \quad (5)$$

in a purely electric field $E^k=F'^{0k}$:

$$\mu F'^{kl}=0, \quad \mu F'^{01}=\frac{e}{s}A_2, \quad \mu F'^{02}=-\frac{e}{s}A_1, \quad \mu F'^{03}=0. \quad (6)$$

Since div **E** in this case is proportional to (curl $\mathbf{A})_3$, a filamentary solenoid with magnetic flux closed within itself becomes a filament carrying a uniformly distributed electric charge. This is in fact an AC configuration.

The results found here are the same as those of Ref. 6. The parameter s acquires a direct physical meaning, of an integral of motion, instead of being a parameter of the representation of the γ matrices. The correspondence between the equations for the AB and AC configurations does indeed turn out to be exact, but only in pure states with a definite spin projection, $s/2$, onto the solenoid (or charged filament). For states which do not have a definite spin projection onto the filament, a switch from AB to AC (or vice versa) changes the nature of the interference of contributions. This circumstance gives rise to the difference between the AB and AC cross sections in Ref. 6 in the case **nz** $\neq \pm 1$.

3. For particles with a larger spin, the situation is even more complicated.

We consider a spin-1 particle. Its behavior in an external electromagnetic field can be described by the Duffin–Kemmer equation[7,8]

$$[\beta^{\mu}(i\partial_{\mu}-eA_{\mu})-m]\Psi=0 \quad (7)$$

for a ten-component column vector Ψ. The 10×10 matrices β^{μ} satisfy the commutation relations

$$\beta^{\mu}\beta^{\nu}\beta^{\sigma}+\beta^{\sigma}\beta^{\nu}\beta^{\mu}=g^{\mu\nu}\beta^{\sigma}+\beta^{\mu}g^{\nu\sigma}. \quad (8)$$

The generators of Lorentzian transformations Ψ are the matrices

$$S_{\mu\nu}=i[\beta_{\mu},\beta_{\nu}], \quad (9)$$

which satisfy the same commutation relations as $(1/2)\sigma_{\mu\nu}$.

We introduce the pseudovector

$$\xi_\mu = \frac{1}{2i} \epsilon_{\mu\nu\sigma\kappa}\beta^\nu\beta^\sigma\beta^\kappa. \tag{10}$$

It satisfies the relations

$$\beta_0\xi_0\beta_0=0, \quad \xi_k\beta_0=\beta_0\xi_k=\epsilon_{k\ell n}S_{\ell n}\beta_0^2, \quad k,\ell,n=1,2,3. \tag{11}$$

Since we have

$$\beta_0\Psi=\Psi,$$

in the proper frame of a particle with a positive energy, ξ_k turns out to be a spin pseudovector for a particle at rest, and the expectation value of ξ_0 vanishes. In the proper frame (as well as in any other frame), ξ_μ thus gives us a spin pseudovector.

For the Duffin–Kemmer equation in the AB configuration, the operator ξ_3 gives us an integral of motion, whose physical meaning is obviously the z component of the spin pseudovector. It takes on three values. By analogy with (3), we introduce the eigenstates

$$\xi_3\Psi_s=s_3\Psi_s, \quad s_3=\pm 1,0. \tag{12}$$

Here ξ_3 has the properties

$$\begin{aligned}\xi_3\beta_1&=\beta_1\xi_3=S_{02}+\text{additional terms},\\ \xi_3\beta_2&=\beta_2\xi_3=-S_{01}+\text{additional terms}.\end{aligned} \tag{13}$$

The additional terms, acting on Ψ_s, which does not depend on z, vanish in the case $s_3 \neq 0$. For $s_3=\pm 1$ we thus have

$$\beta_1\Psi_s=\frac{1}{s_3}\beta_1\xi_3\Psi_s=\frac{1}{s_3}S_{02}\Psi_s, \quad \beta_1\Psi_s=-\frac{1}{s_3}S_{01}\Psi_s. \tag{14}$$

These relations are analogous to Eqs. (4), but we stress that they are not valid at all permissible values of s_3. Using (14), we can rewrite Eq. (7) in the AB configuration ($A_0=A_3=0$, $\partial_3\Psi=0$) as follows:

$$\left(i\beta^\kappa\partial_\kappa-m-\frac{\mu}{2}F'^{\alpha\beta}S_{\alpha\beta}\right)\Psi_s=0. \tag{15}$$

This equation corresponds to an AC configuration. The components $F'^{\alpha\beta}$ are given by (6), with s replaced by s_3.

4. We now consider a particle with arbitrary spin $S>1/2$. Its wave function can be described by a bispinor of rank $2S$, symmetrized with respect to all indices and satisfying the equation

$$[\beta^\mu(i\partial_\mu-eA_\mu)-m]\Psi=0. \tag{16}$$

Here we have

$$\beta_\mu = \frac{1}{2S}\sum_{n=1}^{2S} \gamma^\mu_{(n)}, \tag{17}$$

and each Dirac matrix $\gamma^\mu_{(n)}$ acts on only the nth index of Ψ.

In the case $S=1$, Eq. (16) is the same as the Duffin–Kemmer equation.[7,8] In the case $S=3/2$, it has a structure analogous to that of the Fierz–Pauli equations.[9]

In the AB configuration, an integral of motion again arises, with the meaning of the z component of a spin pseudovector:

$$\xi_3 = \sum_{n=1}^{2S} \left(\frac{1}{2}\gamma^3\gamma^5\right)_{(n)}. \tag{18}$$

This integral can take on the $2S+1$ values from $-S$ to $+S$.

We first consider eigenstates Ψ_s, for which we have

$$\xi_3\Psi_s = sS\Psi_3, \quad s=\pm 1. \tag{19}$$

Obviously, these eigenstates must simultaneously be $(\gamma^3\gamma^5)(n)$ eigenstates for all values of n:

$$(\gamma^3\gamma^5)_{(n)}\Psi_s = s\Psi_s. \tag{20}$$

Using relations like (4), we can rewrite Eq. (16) for Ψ_s, in the AB configuration, as follows:

$$\left(i\beta^\kappa\partial_\kappa - m - \frac{\mu}{2}F'^{\alpha\beta}\frac{1}{S}S_{\alpha\beta}\right)\Psi_s = 0. \tag{21}$$

This equation again describes a neutral particle with an anomalous magnetic moment μ in field (6). Here

$$S_{\kappa\delta} = S^2 i[\beta_\kappa,\beta_\delta] = \sum_{n=1}^{2S}\left(\frac{1}{2}\sigma_{\kappa\delta}\right)_{(n)} \tag{22}$$

are the generators of the Lorentz transformations Ψ.

Under the condition

$$|S_3| < S$$

the eigenstates of the operator ξ_3 are not eigenstates of $(\gamma^3\gamma^5)_{(n)}$, so the AB and AC configurations are not equivalent for them.

5. Let us generalize this discussion, assuming that a particle in an AB configuration has both the charge e and an anomalous magnetic moment μ_0. This particle is described by the equation

$$\left[\beta^\kappa(i\partial_\kappa - eA_\kappa) - m - \frac{\mu_0}{2S}F^{\alpha\beta}S_{\alpha\beta}\right]\Psi = 0 \tag{23}$$

in the AB configuration ($A_0=A_3=0$, $F^{12}=H_3, \partial_3\Psi=0$). Proceeding as before, we again find Eq. (21) for states with $S_3=\pm S$, but now with a different field:

$$\mu F'^{01}=\frac{e}{s}A_2,\quad \mu F'^{02}=-\frac{e}{s}A_1,\quad \mu F'^{03}=0,$$
$$\mu F'^{12}=\mu_0 F^{12},\quad \mu F'^{13}=\mu F'^{23}=0. \tag{24}$$

We have thus again found the equation of a neutral particle with a magnetic moment, but now in the field of a filament which also carries a uniformly distributed charge and which is in a static magnetic field. A configuration of this sort and its topological properties have been discussed previously[10] for a charged spin-1/2 particle.

6. Let us briefly discuss the results. We have shown that an AB configuration is exactly the dual of the AC configuration, not only for spin 1/2, but also for arbitrary spin. This duality, however, holds only for a pure state with a spin projection $S_3=S$ or $S_3=-S$. For pure states with $|S_3|<S$, or for mixed states, duality does not hold. These two situations have a common property: They contain a spin precession, in one form or another. The motion of the particle in these cases is thus not exactly two-dimensional, although the momentum of the particle and the external field are two-dimensional. If we have $S_3=S$ or $S_3=-S$, on the other hand, there is no precession, and the motion is genuinely two-dimensional. It is for motions of this type that the duality of the AB and AC configurations hold. Interestingly, this result agrees with the original paper by Aharonov and Casher,[2] in which a precession of the magnetic moment was not considered. The special role played by the maximum spin projections (maximum in absolute value) is also manifested in the structure of AB scattering amplitudes.[11]

Another interesting property is the substantial difference between the normal and anomalous magnetic moments. The AB motion of a particle which has only a normal moment corresponds to an AC motion in a purely electric field. The incorporation of an anomalous moment in the AB motion requires the addition of a magnetic field in the AC configuration. As a result, the seemingly simple case of a spinor particle with a zero total magnetic moment turns out to be complicated. A charged particle of this sort has both normal and anomalous moments (the anomalous moment cancels the normal moment). This circumstance means that one must use a set of electric and magnetic fields in order to go over to the AC configuration.

The results derived here can be tested experimentally. The existence of the Aharonov–Casher topological effect has been demonstrated[12] for spin 1/2 in experiments with neutrons. The use of polarized beams of neutral atoms in experiments of this sort would make it possible to study higher spins and to see the special role played by the maximum spin projections.

Our interest was attracted to the questions discussed here as the result of a meeting and discussions with A. G. Aronov, to whose memory we dedicate this paper.

We wish to thank the Russian Fund for Fundamental Research for support (Grant 94-02-04039).

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Translated by D. Parsons