Stability of two- and three-dimensional optical solitons in a media with quadratic nonlinearity

S. K. Turitsyn
Institute of Automation and Electrometry, 630090 Novosibirsk, Russia, and Institut für Theoretische Physik I, Heinrich-Heine-Universität Düsseldorf, 40225 Düsseldorf, Germany

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It is demonstrated that solitons existing due to mutual trapping of the fundamental and second-harmonic waves in a dispersive medium with a cascaded second-order nonlinearity are stable. It is proved that solitons realize the minimum of the Hamiltonian for the fixed Manly–Row integral. © *1995 American Institute of Physics.*

Recent progress in studying materials with high second-order nonlinearities (see, e.g., Ref. 1 and the bibliography cited there) stimulates interest in the problem of nonlinear wave propagation in such media. Dielectric materials with quadratic nonlinearity (the so-called χ^2 materials) provide one of the fastest electronic nonlinearities available to date. The possibility of generating in such materials a large, intensity-dependent phase shift is of interest for future all-optical devices.[2] Experimental observation of the cascading process has been reported recently in Ref. 3. In fact, the existence of solitons due to mutual trapping of the fundamental and second-harmonic waves in the media with quadratic nonlinearities has been predicted in Ref. 4. Structure and dynamics of temporal and spatial solitons in the χ^2 materials have been investigated in Refs. 4–8. These studies have revealed the existence of various types of bright and dark solitons.[6–8] The central question concerning the physical relevance of these solitons is their stability.

In this letter we study the stability of multidimensional solitons which exist due to parametric interactions between the fundamental harmonic and the second harmonic in quadratic nonlinear media. We demonstrate that solitons are stable because they realize a minimum of the Hamiltonian for fixed Manly–Row integral for any dimension of the problem.

The spatial-temporal evolution of the slowly varying, dimensionless envelopes of the fundamental U and the second-harmonic V waves is governed by the following basic system written in the Hamiltonian form

$$i\frac{\partial U}{\partial \xi}=-\Delta U-U^*V=\frac{\delta H}{\delta U^*},\quad i\frac{\partial V}{\partial \xi}=-\frac{1}{2}\Delta V-\frac{1}{2}U^2=\frac{\delta H}{\delta V^*}. \tag{1}$$

Here we use the notation of Ref. 6, ξ is the normalized propagation distance, and Δ is the Laplacian. The Hamiltonian is $H=I_1-I_2$, where

$$I_1=\int\left[\left|\nabla U\right|^2+\frac{1}{2}\left|\nabla V\right|^2\right]d\mathbf{r},\quad I_2=\frac{1}{2}\int[U^{*2}V+U^2V^*]d\mathbf{r}.$$

Besides the Hamiltonian, the Manly–Row integral $P=\int[|U|^2+2|V|^2]d\mathbf{r}$ is a conserved quantity.

We study the stability of the two- and three-dimensional, localized, symmetrical, stationary solutions of Eqs. (1) of the form $U=\exp(i\lambda\xi)f(r)$, $V=\exp(i\lambda\xi)g(r)$.

The profile of solitons is given by the nonlinear eigenvalue problem for λ, f, and g.

$$-\lambda f+\Delta f+fg=0, \quad -4\lambda g+\Delta g+f^2=0. \tag{2}$$

This system of equations can be rewritten in a variational form

$$\delta(H+\lambda P)=0. \tag{3}$$

This means that solitons realize the extremum of the Hamiltonian for a fixed P. We will show that ground symmetrical solutions realize a minimum of the H. From Eq. (3) we can directly express the Hamiltonian in terms of P and λ on the soliton solution. Let us consider the trial functions for the variational problem (3) in the form $f_1=\alpha f_{\rm sol}$ and $g_1=\alpha g_{\rm sol}$, where $f_{\rm sol}$ and $g_{\rm sol}$ stand for the ground solutions of Eq. (2). Varying α near 1, we find

$$\left.\frac{\partial}{\partial\alpha}\right|_{\alpha=1}(H+\lambda P)=2I_{1\rm sol}-3I_{2\rm sol}+2\lambda P_{\rm sol}=0. \tag{4}$$

Similar procedure with trial functions of the form $f_2=f_{\rm sol}(\beta r)$ and $g_2=g_{\rm sol}(\beta r)$ give the relation

$$\left.\frac{\partial}{\partial\beta}\right|_{\beta=1}(H+\lambda P)=(d-2)I_{1\rm sol}-dI_{2\rm sol}+d\lambda P_{\rm sol}=0, \tag{5}$$

where d is a dimension of the problem. Straightforward algebraic manipulations yield

$$H_{\rm sol}=-\frac{4-d}{6-d}\lambda P_{\rm sol}\,. \tag{6}$$

Using simple scaling, we can present ground solutions in the form $f_{\rm sol}(r,\lambda)=\lambda f_0(\sqrt{\lambda}r)$, $g_{\rm sol}(r,\lambda)=\lambda g_0(\sqrt{\lambda}r)$. Here we introduce f_0 and g_0 as a ground solution of Eq. (2) with $\lambda=1$. We can now express $P_{\rm sol}$ in terms of $P_0=P_{\rm sol}[f_0,g_0]$:

$$P_{\rm sol}=\lambda^{\frac{4-d}{2}}P_0. \tag{7}$$

The Hamiltonian on the ground solution can be written in the form

$$H_{\rm sol}=-\frac{4-d}{6-d}\left(\frac{P_{\rm sol}}{P_0}\right)^{\frac{2}{4-d}}P_{\rm sol}. \tag{8}$$

To demonstrate that the ground soliton solution realizes the minimum of the Hamiltonian for a fixed P, we need to prove some interpolation estimate for I_2 in terms of I_1 and P. Let us consider the minimization problem for the functional $J[f,g]=P^{(6-d)/4}I_1^{d/4}/I_3$. It can be shown that a minimum is attained on the ground symmetrical soliton solution of Eq. (2). In the proof we follow the procedure used in Ref. 9 (see Ref. 10 for details). The functional J is invariant under the transformation

$\tilde{f}=\nu f(\mu r)$, $\tilde{g}=\nu g(\mu r)$. Thus, by scaling we can take $I_1=d/(6-d)\lambda P$ and $I_3=4/(6-d)\lambda P$. Computing the Euler–Lagrange equation for J, we obtain Eq. (2). Indeed,

$$\frac{6-d}{4}\frac{1}{P}f-\frac{d}{4}\frac{1}{I_1}\Delta f-\frac{1}{I_3}f^*g=0 \tag{9}$$

and

$$4\frac{6-d}{4}\frac{1}{P}g-\frac{d}{4}\frac{1}{I_1}\Delta g-\frac{1}{I_3}f^2=0. \tag{10}$$

Using simple scaling transformation, it is easy to obtain Eq. (2) from these equations. The minimum is attained functions f and g which are positive and functions of r alone. The compactness lemma, provided that such a solution exists, has been proved in Ref. 9.

Thus, we find that the minimum of J is attained at the ground soliton solution, and that it can be calculated as

$$\min(J)=C_0=\left(\frac{d}{6-d}\right)^{d/4}\left(\frac{6-d}{4}\right)P_0^{1/2}. \tag{11}$$

From it we can obtain an interpolation estimate with the "best constant" C_0: $I_3\leqslant C_0P^{(6-d)/4}I_1^{d/4}$. Substituting this estimate into the Hamiltonian, we obtain the lower estimate of H for the fixed P

$$H=I_1-I_3\geqslant I_1-C_0P^{6-d/4}I_1^{\frac{d}{4}}\geqslant P^{\frac{6-d}{4-d}}C_0^{-\frac{4}{4-d}}[(d/4)^{4/(4-d)}-(d/4)^{d/(4-d)}]$$

$$=-\frac{4-d}{6-d}P^{\frac{6-d}{4-d}}P_0^{2/d-4}=H_{\mathrm{sol}}. \tag{12}$$

Thus, the ground symmetrical soliton solution of Eq. (2) realizes the minimum of the Hamiltonian. It is now easy to see that the functional $L=H-H_{\mathrm{sol}}$ satisfies all requirements for the Lyapunov function, and that solitons are stable due to Lyapunov's theorem.

In conclusion, we have shown that solitons due to mutual trapping of the fundamental and second-harmonic waves that propagate in a medium with quadratic nonlinearities are stable. The approach and results obtained can be easily applied to other models of different physical context, where the parametric wave interactions are generated by quandratic nonlinearities.

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