New corrections to hyperfine splitting and lamb shift and the value of the Rydberg constant

M. I. Eides[1)]

Petersburg Nuclear Physics Institute, 188350 Gatchina, St. Petersburg, Russia

V. A. Shelyuto

D. I. Mendeleev Institute of Metrology, 198005 St. Petersburg, Russia

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Corrections to the hyperfine splitting and Lamb shift of order $\alpha^2(Z\alpha)^5$, induced by the diagrams with radiative photon insertions in the electron line, are calculated in the Fried–Yennie gauge. These contributions are equal to $-7.725(1)\alpha^2(Z\alpha)^5/\pi n^3(m_r/m)^3 m$ and $-0.6726(4)\alpha^2(Z\alpha)/\pi n^3 E_F$ for the Lamb shift and the hyperfine splitting, respectively. The phenomenological implications of these results are discussed. © *1995 American Institute of Physics.*

Theoretical work on the high-order corrections to hyperfine splitting (HFS) and Lamb shift was concentrated recently on contributions of order $\alpha^2(Z\alpha)^5$. There are six gauge-invariant sets of diagrams, which produce corrections of order $\alpha^2(Z\alpha)^5$ (Refs. 1 and 2). All contributions induced by the diagrams, which contain closed electron loops, were obtained recently in Refs. 1–5 for the case of hyperfine splitting and in Refs. 2 and 6–9 for the case of the Lamb shift.

We report here the results of our calculation of the contributions of order $\alpha^2(Z\alpha)^5$ to the HFS and Lamb shift induced by the gauge-invariant set of nineteen, topologically different diagrams with insertions of two radiative photons (see Table I).[2)]

For the total correction of order $\alpha^2(Z\alpha)^5$ to the HFS and the Lamb shift produced by all nineteen diagrams with radiative photon insertions in the electron line we obtain

$$\Delta E_{\mathrm{HFS}}^{(r)} = -0.6726(4)\,\frac{\alpha^2(Z\alpha)}{\pi n^3}\,E_F, \tag{1}$$

and

$$\Delta E_L^{(r)} = -7.725(1)\,\frac{\alpha^2(Z\alpha)^5}{\pi n^3}\left(\frac{m_r}{m}\right)^3 m. \tag{2}$$

While this work was in progress, two other papers were published. In those papers the contributions of the same nineteen diagrams to HFS[5] and the Lamb shift[12] were calculated. Despite the great differences in the approaches used in the present work and in Refs. 5–12, the numerical factors in Eq. (1) and Eq. (2) are compatible with $-0.63(4)$ in Ref. 5 and with $-7.61(16)$ in Ref. 12, respectively. Our numbers are about two orders

TABLE I. Corrections to the HFS and Lamb shift.

Diagram	HFS $\frac{\alpha^2(Z\alpha)}{\pi n^3}E_F$	Lamb shift $\frac{\alpha^2(Z\alpha)^5}{\pi n^3}\left(\frac{m_r}{m}\right)^3 m$
a	9/4	0
b	−6.65997(1)	2.9551(1)
c	3.93208(1)	−2.2231(1)
d	−3.903368(79)	−5.238023(56)
e	4.566710(24)	5.056278(81)
f	−3.404163(22)	−1.016145(21)
g	2.684706(26)	−0.1460233(52)
h	33/16	153/80
i	0.054645(46)	−5.51658(54)
j	−7.14937(16)	−7.76813(18)
k	1.465834(20)	1.959589(33)
l	−1.983298(95)	1.74815(38)
m	3.16956(16)	1.87540(17)
n	−3.59566(14)	−1.30584(18)
o	1.804775(46)	−12.06751(47)
p	3.50608(16)	6.13748(30)
q	−0.80380(15)	−7.52525(74)
r	1.05298(18)	14.36733(44)
s	0.277203(27)	−0.930268(72)
Total	−0.6726(4)	−7.725(1)

of magnitude more precise and further improvement of accuracy may be achieved. The reason for the higher accuracy is the use of the FY gauge, where the extrapolation in the photon mass can be avoided.

Numerically the correction to the muonium HFS in the ground state produced by the diagrams under consideration is

$$\Delta E^{(r)}_{\mathrm{HFS}} = -0.3710(2)\ \mathrm{kHz} \tag{3}$$

and the total contribution of order $\alpha^2(Z\alpha)E_F$ is given by

$$\Delta E_{\mathrm{HFS}} = 0.4256(2)\ \mathrm{kHz}. \tag{4}$$

Collecting all theoretical contributions to HFS (see, e.g., the reviews in Refs. 5 and 13) and using for the calculation the value of α from Ref. 14, we obtain the theoretical value for the muonium HFS in the ground state

$$\Delta E_{\mathrm{HFS}} = 4463302.55(0.18)(0.18)(1.33)\ \mathrm{kHz}, \tag{5}$$

where the first error reflects the uncertainty of the fine structure constant, the second is induced by the uncertainty of the contribution of order $\alpha(Z\alpha)^2E_F$, and the third is determined by the experimental error in measuring the electron-muon mass ratio m/M.

TABLE II. $2S_{1/2}-2P_{1/2}$ Lamb shift.

ΔE (kHz)	
1 057 845 (9)	Experimental result, Ref. 26.
1 057 857. 6 (2.1)	Experimental result, Refs. 27 and 28.
1 057 839 (12)	Experimental result, Ref. 29.
1 057 810 (4) (4)	Theory, this work, and r_p=0.805 (11) fm, Ref. 24.
1 057 829 (4) (4)	Theory, this work, and r_p=0.862 (12) fm, Ref. 25.
1 057 854 (16)	Self-consistent $1S$, Ref. 30.
1 057 835 (15)	Self-consistent $1S$, Refs. 31 and 32.
1 057 847 (13)	Self-consistent $1S$, Ref. 33.

The agreement between theory and experiment[15] is excellent. The phenomenological situation and the influence of the result in Eq. (1) on the values of the electron-muon mass ratio and the fine structure constant is discussed in great detail in Refs. 5 and 13.

The case of Lamb shift deserves more comments. Numerically, the corrections to the $1S$ and $2S$ Lamb shifts produced by the nineteen diagrams under consideration are

$$\Delta E_L^{(r)}(1S) = -334.25(4)\ \text{kHz},$$
$$\Delta E_L^{(r)}(2S) = -41.781(5)\ \text{kHz}, \tag{6}$$

while the respective total contributions of order $\alpha^2(Z\alpha)^5 m$ are given by

$$\Delta E_L(1S) = -296.94(4)\ \text{kHz},$$
$$\Delta E_L(2S) = -37.117(5)\ \text{kHz}. \tag{7}$$

Let us now consider briefly the current status of the Lamb shift theory. The theoretical predictions presented below are obtained with the help of the expressions for the Lamb shift contributions, as collected in the reviews of Refs. 16 and 17, amended, besides corrections obtained above and in Ref. 12, with some other recent results.[18–23]

The accuracy of calculations of the Lamb shift intervals is limited by the magnitude of the yet uncalculated contributions of orders $(Z\alpha)^6(m/M)m, \alpha^3(Z\alpha)^4 m$ and $\alpha^2(Z\alpha)^6 m$. Our estimate of the theoretical uncertainty of the expression for the Lamb shift is about 28 kHz for the $1S$ state and about 4 kHz for the $2S$ state.

The other limit on the accuracy of the theoretical calculation of the Lamb shift is set by the experimental value of the proton rms charge radius. There are two contradictory experimental results for this radius.[24,25] The accuracy of the proton rms charge radius claimed by the authors of Refs. 24 and 25 produces an uncertainty of about 32 kHz for the $1S$ state and about 4 kHz for the $2S$ state.

Experimental data for the $2S_{1/2}-2P_{1/2}$ Lamb shift and the results of our theoretical calculations are presented in Table II. The first error of the theoretical values in Table II is determined by the yet uncalculated contributions to the Lamb shift and the second error reflects the experimental uncertainty of the proton rms charge radius. There are two immediate conclusions about the data in Table II. First, as already mentioned in Ref. 12, the results of the proton rms radius measurement in Ref. 24 should be in error, since the

TABLE III. $1S$ Lamb Shift.

ΔE (kHz)	
8 172 860 (60)	Experimental value, Ref. 30.
8 172 815 (70)	Experimental value, Ref. 31.
8 172 844 (55)	Experimental value, Ref. 33.
8 172 915 (129)	Self-consistent value, Ref. 30.
8 172 763 (117)	Self-consistent value, Refs. 31 and 32.
8 172 858 (107)	Self-consistent value, Ref. 33.
8 172 729 (28) (32)	Theory, this work.

respective value of the proton charge radius is clearly inconsistent with all the results of the Lamb shift measurements. Second, we have to reject either the result of the most precise measurement of the $2S_{1/2}-2P_{1/2}$ splitting or the experimental value of the proton charge radius, as measured in Ref. 25, since the Lamb shift value in Ref. 27 contradicts the theoretical value calculated employing the rms radius in Ref. 25 by more than five standard deviations. The results of two other measurements of the classical Lamb shift are compatible with theory, so we will accept below the value of the proton charge radius, as obtained in Ref. 25.

The unbiased extraction of the $1S$ Lamb shift from the experimental data is still a problem. The standard approach consists in adopting one or the other $2S_{1/2}-2P_{1/2}$ experimental result and extracting with its help the value of the $1S$ Lamb shift from the experimental data. All experimental values in Table III for the $1S$ Lamb shift are obtained in this manner with the help of the experimental values in Ref. 26 or in Ref. 29 for the classical Lamb shift. These values should be compared with our theoretical prediction cited in Table III. The results of all experiments mentioned in Table III are consistent and their agreement with the theoretical value is satisfactory.

One can extract self-consistent values of the $1S$ Lamb shift from the experimental data[30,31,33] unambiguously, without reference to the $2S_{1/2}-2P_{1/2}$ experimental results, with the help of the theoretical relation between the $1S$ and $2S$ Lamb shifts

$$8E_L(2S)-E_L(1S)=\Delta, \quad (8)$$

where $\Delta = 187\,234\ (7)$ kHz. The difference $8E_L(2S)-E_L(1S)$ is known theoretically to a higher precision than the values of the $1S$ and $2S$ Lamb shifts themselves (see also the discussion in Ref. 28).

The self-consistent values for the $1S$ Lamb shift in Table III have somewhat larger errors than the "experimental" results in this table. However, they do not depend on the experimental value of the $2S_{1/2}-2P_{1/2}$ Lamb shift and on the value of the proton charge radius. The accuracy of the self-consistent numbers is mainly determined by the accuracy of the frequency measurements in Refs. 30, 31, and 33. A factor of 4–5 reduction of the experimental errors would lead to a self-consistent determination of the $1S$ Lamb shift with the same accuracy as the accuracy of the "experimental" values cited in Table III. One may even invert the usual approach and extract the values of the $2S_{1/2}-2P_{1/2}$ Lamb shift from the respective self-consistent $1S$ values (see the last three lines in Table II).

TABLE IV. Rydberg constant.

R_∞ (cm^{-1})	
109 737. 315 684 1 (42)	Experimental value, Ref. 32.
109 737. 315 683 4 (24)	Experimental value, Ref. 31.
109 737. 315 686 8 (58) (20)	Self-consistent value, Refs. 32 and 30.
109 737. 315 681 1 (52) (14)	Self-consistent value, Refs. 31 and 32.
109 737. 315 679 7 (12) (20) (14)	Theory, this work, and Ref. 32.
109 737. 315 680 2 (05) (14) (06)	Theory, this work, and Ref. 31.

Recent theoretical development opens new ways to a more precise determination of the Rydberg constant, in addition to the one adopted now (see, e.g., Refs. 31 and 32). First, we can use the self-consistent values of the $1S$ and $2S_{1/2}-2P_{1/2}$ Lamb shifts to obtain the value of the Rydberg constant. Today, such an approach leads to a loss of accuracy (see Table IV, where the first error in the self-consistent values is determined by the accuracy of the self-consistent Lamb shift values and the second is determined by the accuracy of the frequency measurement), but greater accuracy may be achieved in the future. Important advantage of such an approach is that the value obtained in this way is independent of the direct experimental results on the $2S_{1/2}-2P_{1/2}$ Lamb shift and independent of the value of the proton charge radius. Second, the new approach simply rejects the experimental data on the Lamb shifts and uses for direct determination of the Rydberg constant the data on the frequencies of the transitions between the levels with different principal quantum numbers. Such an approach is feasible now, since the accuracy of the theoretical formulas for the frequencies of the transitions is determined by the theoretical error of the expression for the $1S$ (or $2S$) Lamb shift which is about 28 kHz (and is even smaller for the $2S$ Lamb shift) and is thus smaller than the experimental error of the frequency determination. The respective values of the Rydberg constant which are derived from independent experimental data[31,32] are presented on the last two lines in Table IV, where the first error is determined by the accuracy of the theoretical expression, the second is defined by the experimental error of the frequency measurement, and the third one is determined by the experimental error in the determination of the proton charge radius. These values are consistent, they are more accurate than those obtained by other methods, and they are the most precise current values. The natural drawback of this approach is the dependence of the obtained value of the Rydberg constant on the proton charge radius.

In conclusion, we would like to emphasize that the high accuracy of the Lamb shift theory opens new perspectives in determining the Rydberg constant and the Lamb shift in the $1S$ and $2S$ states. Four directions of the experimental investigations, namely, the more precise measurement of the transitions between levels with different principal quantum numbers, more precise measurement of the $1S$ and $2S$ Lamb shifts, and the direct measurement of the proton charge radius seem especially promising. All these experiments are mutually complementary, since they lead to the values of the Rydberg constant of comparable accuracy based on the different kinds of experimental data. On the theoretical side, calculation of the unknown corrections to the energy levels, with the goal of reducing the theoretical error in determining the $1S$ Lamb shift to the level of 1 kHz (and, respectively, the $2S$ Lamb shift to several tenth of kHz) seems to be quite promising and feasible.

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[1)]e-mail: eides@lnpi.spb.su

[2)]We have calculated the contributions induced by first nine diagrams earlier.[10,11] A detailed account of our calculations will be presented in a separate publication.

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