Exact solutions in the problem of nonlinear nonlocal diffusion of magnetic flux in films of type-II superconductors

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The problem of the evolution of the spatial distributions of Abrikosov vortices in films of soft and hard type-II superconductors is solved exactly in the flux-conservation regime. The magnetic field in the vortices is perpendicular to the plane of the film. © *1995 American Institute of Physics.*

Let us examine the evolution of magnetic flux in a thin plate or film of a type II superconductor when the magnetic field in the Abrikosov vortices is perpendicular to the plane of the film (see, for example, an experimental study[1]). In this case[2–4] (in contrast with the simpler configuration of the problem in which the magnetic field is parallel to the plane of the plate[5–11]), the motion of each individual Abrikosov vortex is determined by the spatial distribution of all vortices in the system: There is a nonlinear nonlocal diffusion. The equations describing such a diffusion are considerably more complicated than the classical equations of nonlinear diffusion (see, for example, a textbook[12]). We immediately write an equation for the local magnetic induction $B(x,t)$, with which we will be dealing (we will explain below how to derive this equation):

$$\frac{\partial B(x)}{\partial t}=D\,\frac{\partial}{\partial x}\left\{\left|B(x)\right|^{q}\left|\int_{-\infty}^{\infty}dx\,\frac{B(u)}{u-x}\right|^{p}\operatorname{sign}\left(\int_{-\infty}^{\infty}dx\,\frac{B(u)}{u-x}\right)\right\},\tag{1}$$

where D is a dimensional coefficient, and $p>0$ and $q\geqslant 0$ are constants. Here and below, we assume that an infinitely thin plate of a type-II superconductor is oriented perpendicular to the z axis, along which the magnetic induction is directed. The problem is uniform along the y axis. The motion of the vortices occurs along the x axis. The current flows along the y axis. There is no external magnetic field.

In practice, the problem of the evolution of magnetic flux in a film of a type-II superconductor has previously been solved only in the case in which the resistance to the flux is independent of the magnetic field.[2–4] For the most part, linear equations have been studied by numerical methods. We show below that in simple situations involving the evolution of conserved magnetic flux (i.e., $\int_{-\infty}^{\infty}dxB(x,t)=\text{const}$) it is possible to find an exact analytic solution of the problem in what is the most common and most realistic case, in which the resistance to the flux is proportional to B (i.e., the case $q=1$, as we will see below). Completely definite solutions can also be written for other values of q. We will restrict this letter to the most striking case, $q=1$.

Let us outline the derivation of Eq. (1). First, we have ampere's law, which relates $B(x,t)$ and the current density $j(x,t)$ in a film of thickness d (c is the velocity of light):

$$B(x)=\frac{2d}{c}\int_{-\infty}^{\infty}du\,\frac{j(u)}{u-x}. \tag{2}$$

This is a Hilbert transformation,[13] which can easily be inverted:

$$j(x)=-\frac{c}{2\pi^2 d}\int_{-\infty}^{\infty}du\,\frac{Bu}{u-x}. \tag{3}$$

Second, we have the continuity equation for the number density of vortices, ν ($B=\Phi_0\nu$, where Φ_0 is the flux quantum): $\partial\nu/\partial t+\mathrm{div}(\mathbf{J}_v)=0$, where, in the simplest case, the flux of vortices, which is proportional to the electric field, is expressed in terms of the number density of vortices and their velocity by $J_v=|\nu|v$. Here v is the velocity of vortices in which the magnetic field is directed along the z axis, and $-v$ is the velocity of vortices in which the field is antiparallel to the z axis. For a soft superconductor we would have[14] $\eta v=(\Phi_0/c)j$, where the viscosity η is related to the resistivity in the normal phase, ρ_n, and the upper critical field of the superconductor, H_{c2}, by $\Phi_0/\eta=\rho_n c^2/H_{c2}$. As a result, for a soft superconductor, in the specified geometry of the problem, we have

$$\frac{\partial B}{\partial t}=-\frac{\Phi_0}{c\eta}\frac{\partial}{\partial x}\{|B|j\}. \tag{4}$$

(This equation also holds if there is a magnetic field oriented parallel to the surface of a soft superconductor; see Refs. 6–8, where the reasons for the appearance of the absolute value of B are discussed in detail. In those other papers, however, j is proportional to the spatial derivative of B.) Substituting expression (3) for j into (4), we find an equation like (1) for B, with $q=p=1$ and $D=\Phi_0/2\pi^2 d\eta$. The approach taken by Aslamazov and Lempitskiĭ in their well-known paper,[15] for the steady-state situation, is thus being generalized to a dynamic situation.

In the flux-creep regime for hard superconductors it is often legitimate to use the assumption

$$v=l_h\omega_h\left|\frac{j}{j_c}\right|^{U_0/kT}\frac{j}{j_c},$$

where l_h and ω_h are the average length and average frequency of the hops of vortices, and U_0 is a characteristic energy of the barrier pinning (see more details in Ref. 5). As a result, we again use the continuity equation and expression (3). For the evolution of $B(x)$ in a hard superconductor we then find Eq. (1), with $q=1$, $p=1+U_0/kT$, and $D=l_h\omega_h(c/j_c\cdot 2\pi^2 d)^p$. If, before the evolution begins, there is already a uniform distribution of vortices in the sample with a nonzero density $B(x)=B_0$, then we find Eq. (1) with $q=0$ for the deviation from B_0 even for $p>1$. In other words, a "genuine" linearization is not possible for hard superconductors.

Let us assume that a magnetic flux $\Phi=\int dxB(x)>0$ is initially injected into the test sample [$B(x)$ is assumed to be nonzero in a bounded region near $x=0$; we recall that the problem is uniform along the y axis, so all the corresponding quantities are determined per unit length along the y axis]. We seek a solution in the self-similar form $B(x,t) \propto t^{-\beta}h(x/\mathrm{const}t^{\alpha})$. Substituting this relation into (1), and requiring conservation of flux, $\Phi=\int dxB(x)=$ const, i.e., $\alpha=\beta$, we find

$$B(x,t)=\left(\frac{tD}{\Phi}\right)^{-1/(q+p)} h\left(\frac{x}{(\Phi^{q+p-1}Dt)^{1/(q+p)}}\right). \tag{5}$$

The self-similar function $h(x)$ (x is now a dimensionless variable) obeys the equation

$$-\frac{1}{q+p}\frac{d}{dx}(xh)=\frac{d}{dx}\left\{\left|h(x)\right|^{q}\left|\int_{-\infty}^{\infty}dy\,\frac{h(y)}{y-x}\right|^{p}\mathrm{sign}\left(\int_{-\infty}^{\infty}dy\,\frac{h(y)}{y-x}\right)\right\}, \tag{6}$$

which can be integrated. (The reason for this happy circumstance is that the indices α and β are equal in the self-similar solution in the flux-conservation regime.) As a result, for values of x for which the function $h(x)$ is nonzero we find

$$\int_{-\infty}^{\infty}dy\,\frac{h(y)}{y-x}=-\left(\frac{1}{q+p}\right)^{1/p}h^{(1-q)/p}(x)|x|^{1/p}\mathrm{sign}(x). \tag{7}$$

[It turns out that the constant of integration can be omitted, since the function $h(x)$ is assumed to be positive and symmetric with respect to $x=0$.]

We first note that in the region $x\sim 0$ we have

$$h(x)-h(0)=-\frac{1}{\pi}\left(\frac{1}{q+p}\right)^{1/p}h^{(1-q)/p}(0)\cot\frac{\pi}{2p}\,x^{1/p}$$

[this behavior can be found from (7) by making use of the well-known analytic properties of the Cauchy integral[13]]. Further, if $h(x)$ has a tail at infinity, then the asymptotic value of $h(x)$ is proportional to $|x|^{-(p+1)/(1-q)}$ since the left side of (7) has an $O(1/x)$ behavior in the limit $|x|\to\infty$. This result means that for $q\geq 1$ we have $h(x)=0$ if $|x|>x_0$, where x_0 is a positive number. We seek a solution for $q=1$ among such functions. It is convenient to switch from the function $h(x)$, which is nonzero on the interval $(-x_0,x_0)$, to the function $h_1(x)$, which is nonzero for $-1<x<1$,

$$h(x)=x_0^{-1/(1-q-p)}h_1(x/x_0), \tag{8}$$

and to find x_0 at the end of the calculation.

Inverting Hilbert transformation (7) in terms of the relations for functions which are bounded at the ends of a finite interval,[13] we find

$$h_1(x)=\frac{1}{\pi^2}\left(\frac{1}{1+p}\right)^{1/p}\sqrt{1-x^2}\int_{-1}^{1}dy\,\frac{1}{\sqrt{1-y^2}}\,\frac{|y|^{1/p}\mathrm{sign}(y)}{y-x}$$
$$=\frac{1}{\pi^2}\frac{1}{p}\left(\frac{1}{1+p}\right)^{1/p}B\left(\frac{1}{2},\frac{2}{2p}\right)\sqrt{1-x^2}\,{}_2F_1\left(\frac{1}{2}-\frac{1}{2p},1;\frac{3}{2};1-x^2\right). \tag{9}$$

Here $B(\ ,\ )$ is the beta function, and ${}_2F_1(\ ,\ ;\ ;\ )$ the hypergeometric function. Recalling that we must have $\Phi=\int dxB(x)$, we find

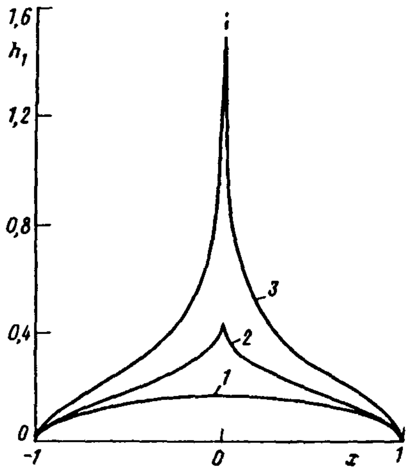

FIG. 1. Self-similar solutions $h_1(x)$ in (9) of Eq. (1) [see also relations (5) and (8)] with $q=1$. These solutions describe the evolution of magnetic flux injected into a type-II superconductor. 1—$p=1$ (soft superconductor); 2—$p=3$; 3—limiting case $p\to\infty$ (the limiting cases for other values of $q<\infty$ are of the same form).

$$x_0=\pi^{p/(1+p)}(1+p)B^{-1/(1+p)}\left(\frac{1}{2},\frac{1}{2p}\right). \tag{10}$$

Equations (5), (8), (9), and (10) constitute the solution of the problem in the $q=1$ case, which is the most interesting one for our purposes.

Figure 1 shows self-similar solutions $h_1(x)$ for $q=1$ and various values of p. In the particular case of a soft superconductor, with $p=q=1$, we have $h_1(x)=\sqrt{1-x^2}/2\pi$ and $x_0=2$. In the limit $p\to\infty$ we find

$$h_1(x)\to(1/\pi^2)\ln\left|\frac{1+(1-x^2)^{1/2}}{x}\right|.$$

The case $x_0\to\pi/2$ describes the propagation of magnetic flux in a hard superconductor at low temperatures. (The latter limit is of a similar form for other finite values of q.) For $q=1$ and arbitrary p, solution (9) near the fronts has a root singularity:

$$h_1(x\sim\pm1)\cong\frac{1}{\pi^2}\frac{1}{p}\left(\frac{1}{1+p}\right)^{1/p}\sqrt{1-x^2}B\left(\frac{1}{2},\frac{1}{2p}\right).$$

Figure 1 shows the extent to which the solutions found here with root fronts and a singularity at the center differ from the solution of the linear equation ($q=0$, $p=1$), in which we have $h(x)\propto 1/(x^2+\pi^2)$. We should point out that spatial distributions of flux which are initially different lead to the self-similar form described here as time elapses. An important point is the condition $\Phi\neq 0$.

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