Multisite antiferromagnetic Ising spin model and universality of the Feigenbaum exponents

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The Feigenbaum exponents α and δ for the tree-site antiferromagnetic Ising spin model on the Husimi tree are calculated. It is shown that the numerical values of these exponents for the real statistical system coincide with the famous universal Feigenbaum exponents with high accuracy. The quantitative description of the transition from order to chaos is also obtained. © *1995 American Institute of Physics.*

1. The concepts Scaling and Universality have played an essential role in the description of statistical systems.[1]

A multisite interaction system on the Husimi tree approximation was recently investigated.[2] First, it was shown that this approach yields a good approximation for the ferromagnetic phase diagrams, which closely match the exact results obtained on a Kagome lattice.[3] Second, a multisite antiferromagnetic interaction was studied and an interesting connection with the area of dynamical systems was made. The qualitative picture of the full doubling bifurcation diagram, including chaos, period-3 windows, etc., for the magnetization of the base site of this system, was exhibited, whereas in the antiferromagnetic Potts model only one period doubling occurred.[4]

On the other hand, it is well known that universality of Feigenbaum exponents applied directly to the period-doubling bifurcation sequence.[5]

Note also that in the anisotropic-next-nearest-neighbor Ising model on a Cayley tree in the infinite coordination limit the existence of chaotic phases associated with strange attractors have been obtained.[6,7]

The aim of our paper is numerical calculation of the Feigenbaum exponents for the three-site antiferromagnetic interaction (TSAI) Ising spin system on the Husimi tree with a finite coordination number and to obtain the quantitative description of the transition from order to chaos for this statistical physical system.

2. The pure Husimi tree,[8] shown in Fig. 1, is characterized γ—the number of triangles which go out of each site. The 0th generation is a single central triangle.

The TSAI model in the magnetic field which is defined by the Hamiltonian

$$H=-J_3'\sum_{\Delta} \sigma_i\sigma_j\sigma_k-h'\sum_{i} \sigma_i, \tag{1}$$

where σ_i takes on values of ± 1, the first sum goes over all triangular faces of the Husimi

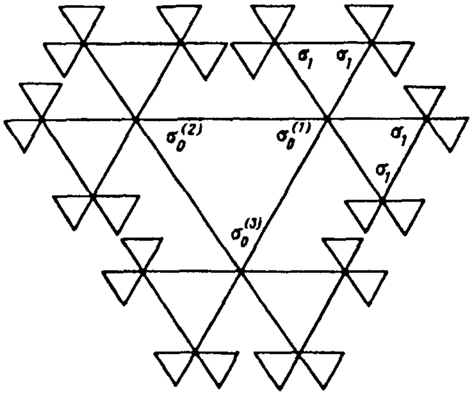

FIG. 1. The Husimi tree with $\gamma=3$.

tree and the second goes over all sites. Additionally, we denote $J_3=\beta J_3'$, $h=\beta h'$, and $\beta=1/kT$, where h is the external magnetic field, T is the temperature of the system, and $J_3<0$ corresponds to the antiferromagnetic case.

The partition function can be written as

$$Z=\sum_{\{\sigma\}} \exp\left\{J_3\sum_{\Delta} \sigma_i\sigma_j\sigma_k+h\sum_{i} \sigma_i\right\}, \tag{2}$$

where the summation goes over all configurations of the system.

The advantage of the Husimi tree which is introduced is that for the models formulated on it, exact recursion relation can be derived. When the Husimi tree is cut apart at the central triangle, it separates into three identical branches, each of which contains $\gamma-1$ branches. The partition function can then be written as follows:

$$Z=\sum_{\{\sigma_0\}} \exp\left\{J_3\sum_{\Delta} \sigma_0^{(1)}\sigma_0^{(2)}\sigma_0^{(3)}+h\sum_{j} \sigma_0^{(j)}\right\}$$
$$\times[g_n(\sigma_0^{(1)})]^{\gamma-1}[g_n(\sigma_0^{(2)})]^{\gamma-1}[g_n(\sigma_0^{(3)})]^{\gamma-1}, \tag{3}$$

where $\sigma_0^{(j)}$ are spins of the central triangle, n is the number of shells, and the equation for one of the branches can be written

$$g_n(\sigma_0)=\sum_{\{\sigma_i\neq\sigma_0\}} \exp\left\{J_3\sum_{\Delta} \sigma_0\sigma_1\sigma_1+h\sum \sigma_1+J_3\sum_{\Delta} \sigma_i\sigma_j\sigma_k+h\sum_{i} \sigma_i\right\}. \tag{4}$$

One of the branches, in turn, can be cut on the site of the 1th generation, which is nearest to the central site. The expression for $g_n(\sigma_0)$ can therefore be rewritten in the form

$$g_n(\sigma_0)= \exp\left\{J_3\sum_{\Delta} \sigma_0\sigma_1\sigma_1+h\sum \sigma_1\right\}[g_{n-1}(\sigma_1^{(1)})]^{\gamma-1}[g_{n-1}(\sigma_1^{(2)})]^{\gamma-1}. \tag{5}$$

From Eq. (5) we easily obtain

$$g_n(+)=e^{J_s+2h}g_{n-1}^{\gamma-1}(+)g_{n-1}^{\gamma-1}(+)+2e^{-J_s}g_{n-1}^{\gamma-1}(+)g_{n-1}^{\gamma-1}(-)$$
$$+e^{J_s-2h}g_{n-1}^{\gamma-1}(-)g_{n-1}^{\gamma-1}(-),$$
$$g_n(-)=e^{-J_s+2h}g_{n-1}^{\gamma-1}(+)g_{n-1}^{\gamma-1}(+)+2e^{J_s}g_{n-1}^{\gamma-1}(+)g_{n-1}^{\gamma-1}(-)$$
$$+e^{-J_s-2h}g_{n-1}^{\gamma-1}(-)g_{n-1}^{\gamma-1}(-).$$

We introduce the following variable:

$$x_n=\frac{g_n(+)}{g_n(-)}. \tag{6}$$

For x_n we can then obtain the recursion relation

$$x_n=f(x_{n-1}), \quad f_{(x)}=\frac{z\mu^2x^{2(\gamma-1)}+2\mu x^{\gamma-1}+z}{\mu^2x^{2(\gamma-1)}+2z\mu x^{\gamma-1}+1}, \tag{7}$$

where $z=e^{2J_s}$, $\mu=e^{2h}$, and $0\leqslant x_n\leqslant 1$. The function $f(x)$ is unimodal: it is continuous, continuously differentiable, and has one maximum x^* in [0,1]. Note that $f(x^*)=1$ for any γ, h, and T. Equation (7) coincides with the equation obtained by Monroe,[2] when the pair interaction is absent.

For magnetization of the central base site we obtain

$$m=\langle\sigma_0\rangle=\frac{e^hg_n^\gamma(+)-e^{-h}g_n^\gamma(-)}{e^hg_n^\gamma(+)+e^{-h}g_n^\gamma(-)}=\frac{e^hx_n^\gamma-1}{e^hx_n^\gamma+1}. \tag{8}$$

3. As was mentioned above, the TSAI system is the nonlinear dynamic system and the qualitative picture of the full doubling bifurcation diagrams, chaos etc., for the magnetization of the base site was shown in Ref. 2.

The questions we wish to address in this paper are how to calculate the Feigenbaum exponents for the TSAI system and whether the calculated values coincide with the famous universal Feigenbaum exponents:

$$\alpha=2.50290\ldots, \quad \delta=4.669201\ldots. \tag{9}$$

Feigenbaum observed for the logistic map (see Ref. 9) two kinds of scaling: that the length 2^n cycle first appears at an r_n value, which satisfies

$$r_n=r_\infty-\text{ const }\delta^{-n}, \quad n\gg 1, \tag{10}$$

where r_∞ is the value of r from which the chaotic behavior ensues, and that the sequence essentially never repeats itself.

The other scaling was a special behavior which occurred near the x^* value for which the map is extremal ($x^*=1/2$ in the logistic map). If we start with the value for x^*, then

$$-\alpha=\frac{d_n}{d_{n+1}}, \quad n\gg 1, \tag{11}$$

where

$$d_n=f_{R_n}^{2^{n-1}}(x^*)-x^*. \tag{12}$$

TABLE I.

Period doubling	H_n	α	δ	Magnetization, m	x_n
$2^1=2$	0.18354515...			−0.6782977	0.5423560
				0.09151673	0.999999
$2^2=4$	0.28692571...	4.86428158...	3.50752342...		
$2^3=8$	0.31861247...	2.19505287...	4.32097441...	−0.8924331	0.3457500
				−0.1373881	0.8200590
				−0.9581313	0.2495890
				0.1182513	0.9733620
				−0.8506408	0.3886090
				−0.2286943	0.7699640
				−0.9640809	0.2369180
				0.1579719	0.9999
$2^4=16$	0.32607381...	2.76000232...	4.5870529...		
$2^5=32$	0.32770666...	2.42990139...	4.65118547...		
$2^6=64$	0.32805801...	2.5381186...	4.66503325		
$2^7=128$	0.32813334...	2.4897987...	4.66830065...		
$2^8=256$	0.32814947...	2.5099532...			
...	...	...	...	...	...
$2^\infty=\infty$	0.3281538...				

In Eq. (12) the R_n are the values of r $(r_1<R_1<...<R_n<r_n)$, and

$$f_{R_n}^{2^n}(x^*)=x^*. \tag{13}$$

Note that the values of R_n and r_n have the same scale, and $r_\infty=R_\infty$. Therefore,

$$R_\infty=R_n-\text{ const }\delta^{-n}. \tag{14}$$

Equations (11) and (14) define two Feigenbaum exponents, which turn out to be "universal."

Now let us turn to our questions. We see from Eq. (7) that the above-mentioned parameter r for the TSAI system on the Husimi tree for each fixed temperature represents the external magnetic field h. The recursion function of Eq. (7) has one maximum at $x^*=1/^{\gamma-1}\sqrt{\mu}$. Note that this x^* depends on the values of T and h, whereas in case of the logistic map it is a constant. Further, we numerically solve Eq. (13) and find the values of H_n (H_n is the analog of R_n). Using these values of H_n and Eqs. (11) and (14), we calculate the Feigenbaum exponents for the TSAI system. All our numerical calculations are done for $\gamma=3$, $T=0.3$, and $J_3=-1$. The results are presented in Table I.

For the constant presented in Eq. (14), which depends on the family of reflection functions,[10] for the TSAI system we obtain the following value: const=0.99..., whereas for the logistic map it is 0.12... .

Using the values of H_n and x_1, x_2,...x_n corresponding to them, we calculate the magnetization for the base site (for each cycle of the period doubling) of this system by using Eq. (8). This means that each 2^n period doubling has n values of the magnetization, which can be explained as an arising of the n-sublattice phase such that $x_1, x_2, \ldots x_n$ determine the states on each sublattice.

We see from the numerical results that for a real statistical physical system the obtained values of α and δ coincide with the famous Feigenbaum exponents [Eq. (9)] with high accuracy, and there confirm their universality again.

The same Feigenbaum exponents for all temperatures below T^* are also calculated with high accuracy. $T^* \approx 0.55$ is the upper bound of the chaotic temperature for the TSAI system when $\gamma=3$, $J_3=-1$.

It is interesting to note that if we set $\gamma=2$ in Eq. (7), rather than $\gamma=3$, then the above-mentioned situation changes dramatically.

Let us consider the system of equations

$$\begin{cases} f(x)-x=0 \\ f'(x)=-1 \end{cases}. \tag{15}$$

Equations (15) determine the point at which the first doubling bifurcation begins.

For the recursion function, when $\gamma=2$, Eq. (15) has the form

$$\begin{cases} \mu^2x^3+z\mu(2-\mu)x^2+(1-2\mu)x-z=0 \\ \mu^2x^2-2z\mu^2x-(1+2\mu)=0 \end{cases}, \tag{16}$$

which for any T and h have only nonphysical solutions. Therefore, for the TSAI system when $\gamma=2$ the period-doubling bifurcations picture is absent. This means that for this statistical physical system there is no second-order phase transition when $\gamma=2$.

4. In this paper we have investigated the TSAI Ising spin model by approximating it with the Husimi tree structures and calculated the Feigenbaum exponents α and δ. The numerical results show that the values of these exponents for a real physical system coincide with the famous universal Feigenbaum exponents with high accuracy. The quantitative description of the transition from order to chaos is also obtained. We see many of the extensively studied and by now familiar properties of the dynamical system theory, which make it possible to study the statistical physical systems in a new context and in a simple manner. In particular, using the well-known technique for dynamical systems, we analytically show that for the TSAI system there is no second-order phase transition when $\gamma=2$.

We believe that the results which we obtained are interesting and we plan to continue to investigate this line of approach for the TSAI system and for several other systems.

On the other hand, study of the chaotic statistical physical system has opened new challenges for theories of stochastic processes, especially in the direction of stochasticity of vacuum in QCD.[11] Interesting results for the $Z(Q)$ gauge model on generalized Bethe lattice was obtained in Ref. 12. It gives reason to assume that the TSAI Ising spin model on the Husimi tree approximation can be connected via the duality[13] with the n plaquette representation of the gauge theory.

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