Acoustic turbulence in media with two types of sound

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The state of wave turbulence in media with two acoustic branches is analyzed for the case in which sound of the high-frequency branch decays into two sound waves of the low-frequency branch. The forward cascade is accompanied by an inverse cascade in this situation. As a result, a power-law turbulence distribution is established at small values of k. The relaxation to a steady-state distribution is also analyzed. When the sound velocities on the different branches are greatly different, the kinetic equations can be simplified substantially through expansion in a small parameter. A steady-state regime is established over an infinite time at small values of k, and over a finite time at large values of k, under the assumption that the spectra are established in a self-similar way.

In certain physical problems it becomes necessary to describe acoustic turbulence in media which have two sound branches: a high-frequency branch with a sound velocity c_1 and a low-frequency branch with a velocity c_2 ($c_1 > c_2$; Ref. 1, for example). In general, many types of nonlinear interactions between the waves must be taken into consideration. In addition to the decays of sound vibrations into sound waves belonging to the same branch, there can be a decay of high-frequency sound into two low-frequency waves, and there can also be Čerenkov radiation of low-frequency sound by high-frequency sound.

Under certain specific conditions, however, it may turn out that only one process plays a major role. In this letter we examine the case in which the first of these nonlinear interactions between different acoustic branches is the governing interaction.

The other possible situation, in which the Čerenkov process plays the major role in the establishment of the turbulence regime, was studied in Ref. 2.

An interesting case is that in which the sound velocities of the different acoustic branches are very different, i.e., the case in which the parameter $\gamma = c_2/c_1$ satisfies the inequality $\gamma \ll 1$. As we will see below, when this parameter is small, one can make much progress toward describing time-varying turbulence processes (including the formation of nonequilibrium distributions). We assume below that the condition $\gamma \ll 1$ holds.

Some real physical examples are superfluid helium in the roton temperature region[3,4] ($T \gtrsim 1$ K) and a plasma with heavy-ion impurities[5] at certain levels of the rate at which energy is pumped into these systems.

In the situation which we are interested in the present letter, power-law turbulence distributions

$$N_{\mathbf{k}}=Ak^s, \quad n_{\mathbf{k}}=Bk^s, \tag{1}$$

can be established. Here $N_{\mathbf{k}}$ and $n_{\mathbf{k}}$ are the occupation numbers for the high- and low-frequency modes. These numbers depend on the wave vector **k**. In addition, A and B are the amplitudes of distributions, and s takes on values from the set $\{-9/2,-4,-1,0\}$ (Ref. 1).

As was shown in Ref. 3, spectrum (1) with the exponent $s=-9/2$ corresponds to the formation of a **k**-independent energy flux $\mathscr{E}$ into the short-wave region:

$$\mathscr{E}=\int d\mathbf{k}\Omega_{\mathbf{k}}N_{\mathbf{k}}+\int d\mathbf{k}\omega_{\mathbf{k}}n_{\mathbf{k}}, \tag{2}$$

where $\Omega_{\mathbf{k}}$ and $\omega_{\mathbf{k}}$ are the dispersion relations for the high- and low-frequency branches. A spectrum with an exponent $s=-1$ is the thermodynamic-equilibrium spectrum.

Let us examine distributions (1) for two other values of s. We first consider the case $s=-4$. Writing the corresponding kinetic equations[1,3] in divergence form (by analogy with the form used in Ref. 6), we can show that the onset of a distribution of this sort stems from the presence of an additional integral of the kinetic equations, $\mathscr{N}$:

$$\mathscr{N}=\int d\mathbf{k}(2N_{\mathbf{k}}+n_{\mathbf{k}}). \tag{3}$$

The flux (Q) of this integral in k space turns out to be negative: $Q<0$. Such a distribution is thus formed asymptotically as $k\to 0$. An estimate of the amplitudes A and B of the spectrum yields $A,B \propto \sqrt{-Q}$.

Distribution (1) with occupation numbers which are independent of k (the case $s=0$) corresponds to the equilibrium situation, as is easily verified. The total equilibrium distribution in the isotropic case is given by

$$N_{\mathbf{k}}=\frac{T}{\Omega_{\mathbf{k}}+\mu}, \quad n_{\mathbf{k}}=\frac{T}{\omega_{\mathbf{k}}+\mu'} \tag{4}$$

and is characterized by a particular temperature T and chemical potentials μ and $\mu'=2\mu$. In the limit $\Omega_{\mathbf{k}},\omega_{\mathbf{k}}\ll\mu$, distribution (4) becomes (1) with the amplitudes $A=T/\mu$ and $B=T/\mu'=T/2\mu$. A spectrum with $s=0$ is thus realized at small values of k in the absence of steady-state fluxes of the integrals in k space.

We turn now to the formation of the turbulence distribution. Making use of the small parameter γ, we can simplify the description of the time-varying processes within the framework of kinetic equations for waves. An estimate of the collision integrals on the right sides of these equations shows that the time scale τ_1, for the change in the distribution of the high-frequency sound, is short in comparison with τ_2, for the low-frequency sound, to the extent that γ is small: $\tau_1/\tau_2\sim\gamma^3$. Accordingly, the evolution of the turbulence distribution at times $t\gg\tau_1$ can thus be described by a kinetic equation for the function $n_{\mathbf{k}}$ from which the distribution $N_{\mathbf{k}}$ has been eliminated (the latter distribution plays the role of a fast variable; this is the adiabatic approximation). The resulting equation is

$$\frac{\partial n_{\mathbf{k}}}{\partial t}=\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}[(n_{\mathbf{k}}+n_{\mathbf{k}_1})n_{\mathbf{k}_2}n_{\mathbf{k}_3}-(n_{\mathbf{k}_2}+n_{\mathbf{k}_3})n_{\mathbf{k}}n_{\mathbf{k}_1}]. \tag{5}$$

The probability $W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ is given by

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}=\int d\mathbf{k}_4 \frac{U_{\mathbf{k}_4\mathbf{k}\mathbf{k}_1}U_{\mathbf{k}_4\mathbf{k}_2\mathbf{k}_3}}{\int d\mathbf{k}_5 d\mathbf{k}_6 U_{\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6}(n_{\mathbf{k}_5}+n_{\mathbf{k}_6})},$$

$$U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}=\pi|V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta(\Omega_{\mathbf{k}}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}),$$

where $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ is the amplitude of the nonlinear process which we are taking into account. Equation (5) has the form of a four-wave kinetic equation written for the case of one acoustic mode.[7] The onset of a quasiequilibrium distribution of the high-frequency sound mode thus leads to the onset of an effective four-wave interaction of waves of the other branch.

Equation (5) has the integral of motion $\mathcal{N}_2=\int d\mathbf{k}n_{\mathbf{k}}$. It follows from this circumstance and from the presence of integral (3), which is an exact property of the original kinetic equations, that the quantities $\mathcal{N}_2$ and $\mathcal{N}_1=(\mathcal{N}-\mathcal{N}_2)/2$ are separately conserved in the adiabatic approximation. Equation (5) can be simplified even further. Since the sound-velocity ratio γ is small, the interaction of the waves in the case of a decay process is local in k space. It follows from conservation of the wave vector and the frequency that the high-frequency wave, with a vector $\mathbf{k}$, splits into two low-frequency waves, whose wave vectors $\mathbf{k}'$ and $\mathbf{k}''$ are such that $|\mathbf{k}|'\approx|\mathbf{k}|''\approx|\mathbf{k}|/2\gamma$ holds.[4] We can approximate the right side of Eq. (5) by a differential operator. For simplicity we restrict the discussion to the case of an isotropic distribution. Expanding the integral in (5) in a series in γ, and retaining only the leading order, we find the equation

$$\frac{\partial n_k}{\partial t}=\Lambda\frac{1}{k^2}\frac{\partial^2}{\partial k^2}k^{11}n_k^3\frac{\partial^2}{\partial k^2}n_k^{-1}, \tag{6}$$

where the positive dimensional constant Λ depends on the thermodynamic properties of the medium. At the same accuracy level, the quasiequilibrium distribution of the high-frequency sound is described at each instant by

$$N_k=n_{k'}/2,\quad k'=k/2\gamma. \tag{7}$$

We find a condition for the applicability of Eqs. (6) and (7) from the condition that the terms omitted from the expansion be small. This condition is

$$\gamma k\frac{\partial n_k}{\partial k}\ll n_k.$$

Equations (6) and (7) are therefore applicable if the logarithmic derivative of the function n_k with respect to $x=\ln k$ is small in comparison with γ^{-1}.

Equations (6) and (7) can be used to describe the formation of steady-state turbulence distributions. Let us examine the onset of asymptotic distributions (1) with $s=-4$ as $k\to 0$ and $s=-9/2$ as $k\to\infty$. It is customary to assume that the development of the turbulence becomes self-similar far from the pump scale, k_p. A self-similar substitution for n_k leads to the following distributions, which depend on the time and which become (1) in the limit $k \to k_p$:

$$n_k(t) = t^4 f(kt), \quad k < k_p, \quad n_k(t) = \tau^9 g(k t^2), \quad k > k_p. \tag{8}$$

Here we have $\tau = t_0 - t$, and t_0 is the time scale for the formation of the short-wave asymptotic form of the distribution, whose magnitude can in principle be determined from the solution of Eq. (6). The functions $f(x)$ and $g(x)$ have the properties

$$f(x) \propto x^{-4}, \quad x \to \infty, \quad f(x) \to 0, \quad x \to 0,$$
$$g(x) \propto x^{-9/2}, \quad x \to 0, \quad g(x) \to 0, \quad x \to \infty. \tag{9}$$

It follows from (8) that an infinitely long time is required for the system to reach a steady state in the limit $k \to 0$. In a real experiment, the inertial interval, within which distribution (1) holds, is bounded from below by the reciprocal linear dimension of the system, $k_L \sim L^{-1}$. The time t_l, over which the boundary of distribution (8) at $k < k_p$ reaches the scale k_l, is given in order of magnitude by

$$t_l \sim \frac{k_p}{k_L} \tau_2,$$

where τ_2 is the time scale for the change in the distribution in the low-frequency mode, estimated from the kinetic equations.

The time required for the formation of the short-wave part of the distribution is finite according to (8).

A difference of this sort in the processes by which nonequilibrium turbulence distributions are established can be explained in the following way. An estimate of the energy $\mathscr{E}$ of distribution (1) shows that the corresponding integrals diverge as $k \to 0$, for both $s = -4$ and $s = -9/2$. Consequently, the small-k region has most of the energy. The establishment of a spectrum $N_{\mathbf{k}}$, $n_{\mathbf{k}} \propto k^{-9/2}$ as $k \to 0$ requires the redistribution of a finite amount of energy, and it takes a finite time. The formation of the asymptotic spectrum $N_{\mathbf{k}}$, $n_{\mathbf{k}} \propto k^{-4}$ as $k \to 0$ leads to the transfer of formally an infinite amount of energy into the long-wave region, and it takes an infinite time.

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