Can a gravitating superfluid liquid rotate?

A. Yu. Andreev, D. A. Kirzhnits, and S. N. Yudin
P. N. Lebedev Physics Institute, Russian Academy of Sciences, 117924 Moscow, Russia

(Submitted 17 April 1995)
Pis'ma Zh. Éksp. Teor. Fiz. **61**, No. 10, 825–829 (25 May 1995)

The question in the title of this paper is answered unambiguously. The rotation of an external object would, by entraining the condensate along with the inertial frame of reference, lead to the formation of a meniscus (the velocity v_s^3 is not zero). This rotation of the condensate, however, would not generate Lense–Thirring forces and would not contribute to the angular momentum of the system (the velocity v_{s3} is zero). In this sense, the condensate, like superfluid ^{4}He at a small angular velocity, cannot rotate.

The huge relaxation time of the period of a pulsar after collapse implies an anomalously weak coupling between the crust and core of the pulsar. It is explained in terms of a superfluidity of the core under the implicit assumption that mechanisms for crust–core coupling other than viscosity are weak.[1] Among these other mechanisms is an effect on the core of Lense–Thirring forces[2,3] generated by rotation of the crust. Before this mechanism is analyzed, it is necessary to solve the more general problem, which goes beyond the field of pulsar physics, of whether it is possible for a heavy superfluid liquid to rotate. This question is the topic of the present letter.

For simplicity we adopt some restrictions: The problem is a steady-state problem. The system is axisymmetric. The rotation is slow (there are no vortex filaments). The order parameter is a scalar. The matter is nonrelativistic (the pressure is much lower than the energy density). We use a system of units with $\hbar = c = 1$.

1. It would seem at first glance that this question would elicit contradictory answers. On the one hand, for the liquid to rotate we would need an "elongated" (in the gauge-theory sense) gradient of the order parameter in the expression for the superfluid current. This is the situation in a superconductor in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, where the gradient $\nabla - ie\mathbf{A}$ is "elongated" and where the London relation holds ($\mathbf{v}_s$ is the superfluid velocity, and e and m are the charge and mass):

$$\nabla \times \mathbf{v}_s = -e\mathbf{B}/m, \quad \mathbf{v}_s = -e\mathbf{A}/m \quad (\mathrm{div}\mathbf{A} = 0). \tag{1}$$

In contrast, in ^{4}He, in which the gradient is "short," we have $\nabla \times \mathbf{v}_s = 0$, and rotation is impossible.[4] It would seem that a heavy liquid in a gravitational field should behave in the same way: In the general theory of relativity, an "elongation" means a transition to a covariant derivative, which has the same effect as an ordinary derivative on a scalar order parameter $\psi \propto \exp(i\alpha)$. Specifically,[5] the expressions for the scalar 4-current and the scalar 4-velocity in the general theory,

$$j_i \propto i[\partial_i \bar{\psi}\psi - \bar{\psi}\partial_i \psi], \quad u_i = \partial_i \alpha, \tag{2}$$

directly imply that the flow is of a potential nature.

2. On the other hand, there is far-reaching similarity between the equations of the general theory of relativity for a weak field and the equations of electrodynamics. This similarity is manifested for the transverse field components of interest here when we make the substitutions[6]

$$m\mathbf{g} \rightleftharpoons e\mathbf{A}, \quad e^2 \rightleftharpoons -4Gm^2, \tag{3}$$

where g is the metric tensor, $g_\alpha = -g_{0\alpha}$ ($\alpha = 1,2,3$) is the gravimagnetic potential, and G is Newton's gravitational constant. In particular, there exists an analog of the familiar equation for $\mathbf{A}$ ($\kappa^2 = 16\pi G\rho$, where ρ is the density):

$$\Delta \mathbf{g} = \kappa^2 \mathbf{v}. \tag{4}$$

The quantity on the right side of this equation is the sum of the superfluid source (s) and the normal source (n). Using the relation[7]

$$\mathbf{v}_s = -\mathbf{g}, \tag{1a}$$

which follows from (1) and (3), we find gravitational analogs of the London equation,[4]

$$(\Delta + \kappa_s^2)\mathbf{g} = \kappa_n^2 \mathbf{v}_n, \tag{4a}$$

and of the corresponding effects of the electrodynamics of superconductors: a rotation of the condensate due to rotation of the normal matter (an analog of the London current) and the formation of a secondary gravimagnetic potential due to rotation of the condensate (an analog of the Meissner effect).[1)]

3. The contradictory situation which arises here contains some obscure points, regarding the meaning of the velocity vector (the covariant or contravariant component) and regarding the actual improvement in accuracy in (4a) (the second term on the left side gives rise to a term of second order in G). The simplest way to clarify these points is to go over to a systematic general-relativity analysis: We assume that only the rotational quantities are small in the exact equations of the general theory of relativity, and as the background metric we adopt the Schwarzschild metric for a spherical object which is at rest.[2] In terms of the spherical coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, with polar axis along the rotation axis, quantities with the spatial index 3 are nonzero. There is no dependence on x^3. In particular, the contravariant component of the 3-velocity is nonzero:[2]

$$v^3 = \dot{x}^3 h^{-1/2}, \quad h = g_{00}. \tag{5}$$

For rigid-body rotation of normal matter at an angular velocity Ω ($x^3 = x_0^3 + \Omega t$), this velocity component is equal to

$$v_n^3 = \Omega h^{-1/2}. \tag{5a}$$

We introduce components of the 4-velocity:[2]

$$u^i = (h^{-1/2}, 0, 0, v^3), \quad u_i = (h^{1/2}, 0, 0, v_3) = g_{ik} u^k.$$

According to (2) and the discussion above, we have

$$v_{s3} = g_{3k} u_s^k = d\alpha/dx^3 = 0. \tag{6}$$

We thus find a generalization of (1a) to the case of a curved space:

$$v_s^3=-g^3h^{-1/2}, \tag{1b}$$

where we have introduced a contravariant component of the 3-vector g_α,

$$g^3=-g^{33}g_3=g_{03}/g_{33}.$$

We thus see that the resolution (at least at a formal level) of the contradiction is that the condensate is immobile in the sense that the covariant component of the 4-velocity is zero [see (6)], while it rotates in the sense that its contravariant component is nonzero [see (1b) and (5)]. In the case $g_{03}\neq 0$, for example, in the field of a rotating object, one of these components may vanish while the other does not. The physical meaning of these arguments is clarified below.

4. Equation (6) has several clear consequences. First, the source of the gravimagnetic potential is a rotation of only the normal component, not the superfluid component: The superfluid source disappears from the Einstein equation $R_{03}-\frac{1}{2}g_{03}R=\frac{1}{2}\kappa^2u_0u_3$. Making use of properties of the Schwarzschild solution, we find the following generalization of Eq. (4a) to the case of a curve space:

$$\left[\beta\left(\partial_r^2+\frac{4}{r}\,\partial_r\right)-\frac{1}{4}\,(\kappa_n^2+\kappa_s^2)r\partial_r-\kappa_n^2\right]g^3=\kappa_n^2\Omega, \tag{4b}$$

where

$$\beta=-g_{11}^{-1}=1-\frac{1}{2r}\int_0^r drr^2(\kappa_n^2+\kappa_n^2).$$

[The superfluid source on the right side of (4) has been cancelled by a geometric term which arises on the left side.] Actually, the condensate, by changing the Schwarzschild metric, manifests only its mass, not its velocity. Accordingly, there is actually no secondary gravimagnetic potential, there is no analog of the Meissner effect, and there is no tachyon instability[1)] in the latter case (according to our preliminary estimates).

Another consequence of (6) is that the condensate does not contribute to the angular momentum of the system, $\mathscr{M}$, which is determined by the asymptotic behavior of the gravimagnetic potential far from the object,[2]

$$g^3\to -2G\mathscr{M}/r^3.$$

This point can also be seen directly from the agreement of the expression $\mathscr{M}_s=-\partial s/\partial x^3$ (s is the action) with the momentum $p_{s3}=mv_{s3}=0$.

To illustrate this discussion, we use the problem of a system with superfluid core (of mass M_s and radius R) with a thin normal crust (of mass M_n). We assume that the densities $\rho_{s,n}$ are constants. The value of g^3 in the core also turns out to be a constant. Using the notation $\lambda=r_g/R$, $r_g=2GM$,

$$\theta=(1-\sigma)/(1-\sigma/4),\quad \sigma=[1-\lambda_n/(1-\lambda_s)]^{1/2},$$

we find the following expression for the angular momentum from Eq. (4b):

$$\mathscr{M}=\theta R^3\Omega/2G.$$

For the effective angular velocity of the condensate we find[2)]

$$\Omega_{\mathrm{eff}}/\Omega = v_s^3/v_r^3 = g^3 = \theta.$$

Under the conditions $\lambda_n \ll \lambda_s$,

$$\mathcal{M} = \tfrac{2}{3} M_n R^2 \Omega/(1-\lambda_s), \quad \Omega_{\mathrm{eff}}/\Omega = \tfrac{2}{3} \lambda_n/(1-\lambda_s),$$

the denominator in these expressions has a purely geometry meaning [it is associated with the value of β in (4b), which appears in the expression for the Laplacian in a curved space].

5. The fact that the velocity in (1b) is nonzero has no effect on quantities which are linear in the angular velocity—the gravimagnetic potential and the angular momentum—but it is manifested in second order in Ω, in the formation of a meniscus at the free surface of the liquid. This conclusion follows from the condition that the pressure p be constant along the surface and from the Bernoulli equation (here[3)] $v^2 = g_{\alpha\beta} v^\alpha v^\beta$):

$$v^2/2 + p/\epsilon + \chi = \mathrm{const}, \quad h = 1 + 2\chi + \dots . \tag{7}$$

This equation is derived from the hydrodynamic equations of the general theory of relativity,[10]

$$w u^k D_k u_i = (\partial_i - u_i u^k \partial_k) p, \quad w = \epsilon + p, \quad d(w/\rho) = dp/\rho,$$

in the steady state and under condition (6). In general, it is of the form

$$h w/\rho \sqrt{1-v^2} = \mathrm{const}.$$

For a weak field, for slow rotation, and for a nonrelativistic medium, we then find (7).

Here we run into the question of a transformation to a frame of reference which is rotating at an angular velocity ω around the same axis (the primed quantities refer to this frame of reference). Using $x^3 = x^{3'} + \omega t'$, $t = t'$, we find the following result for the metric tensor:

$$h' = h + \omega^2 g_{33}, \quad g^{3'} = g^3 + \omega, \quad g'_{33} = g_{33} = -r^2 \sin^2\theta. \tag{8}$$

The increments in h correspond to centrifugal acceleration, while the increments in g^3 correspond to a Coriolis acceleration.

For the transformation of 4-velocities we find

$$u^{0'} = h^{-1/2}, \quad v^{3'} = v^3 - \omega h^{-1/2}; \quad u'_0 = h^{1/2} + \omega v_3, \quad v'_3 = v_3. \tag{9}$$

6. The assertions in Secs. 4 and 5 of this letter can be explained by citing the entrainment of an inertial frame of reference by a massive rotating object,[3] in which the Lense–Thirring acceleration is cancelled by a Coriolis acceleration, with the result that we have $g^{3'} = 0$ [see (8)]. In such a system we have $v_s^{3'} = 0$ and, simultaneously, $v'_{s3} = 0$ according to (1b), (8), and (9). From the standpoint of the velocity v_s^3, the condensate is thus at rest in the inertial frame, meaning that there are no dynamic manifestations of a rotation in the system itself. Actually, this conclusion applies to all frames of reference: In any frame we have $v_{s3} = 0$ by virtue of the last equation in (9) (see Ref. 11, for example, regarding the absence of a contribution to the angular momentum).

The meniscus, which is an effect of second order in Ω, is not one of the dynamic manifestations which we just mentioned. The reason is that in this order the rotating frame of reference under consideration is no longer an inertial frame: It can be shown that an acceleration of the Lense–Thirring type of second order in Ω cannot cancel the centrifugal acceleration, having the same sign as the second term in h' [see (8)].

We wish to thank V. L. Ginzburg and the participants of a seminar which he led, especially V. S. Beskin and V. B. Braginskiĭ. We also wish to thank G. S. Bisnovatyĭ-Kogan and D. M. Sedrakyan for useful discussions.

[1)]This similarity is disrupted by the "incorrect" sign of the second term on the left side of (4a) [this is a consequence of the minus sign in (3); like charges, i.e., masses, attract each other in gravitation]. This circumstance is seen in the onset of an entrainment of the condensate instead of a countercurrent of electrons (Lenz's law); instead of an anomalous diamagnetism we find an anomalous paramagnetism (Chap. 9 in Ref. 8). Furthermore, the spectrum of rotational excitations acquires a tachyon nature, $\omega^2 = k^2 - \kappa_s^2$, which might lead to a "self-unwinding" of the system (a rotational analog of the Jeans instability).

[2)]The quantity $\Omega_{\rm eff}$ for the case $M_s = 0$ has been calculated previously.[3,9]

[3)]The quantities $v_{s\alpha}$ [see (6)] and $g_{\alpha\beta} v_s^\beta$ should not be confused with each other: The former is the covariant component of a 4-vector, while the second is that of a 3-vector.

[1] R. Manchester and J. Taylor, *Pulsars* (Freeman, San Francisco, 1977).
[2] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* [in Russian] (Nauka, Moscow, 1988).
[3] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[4] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1980).
[5] A. A. Grib, S. G. Mamaev, and V. M. Mostepanenko, *Quantum Effects in Intense External Fields* [in Russian] (Atomizdat, Moscow, 1980).
[6] C. Moller, *Theory of Relativity* (Oxford Univ. Press, London, 1975).
[7] B. S. de Witt, Phys. Rev. Lett. **16**, 1092 (1966).
[8] *The Problem of High-Temperature Superconductivity* [in Russian], ed. by V. L Ginzburg and D. A. Kirzhnits (Nauka, Moscow, 1977).
[9] J. M. Cohen, in *Relativity Theory and Astrophysics. 1*, ed. by J. Ehlers (1967), p. 200.
[10] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).
[11] *Black Holes: The Membrance Approach*, ed. by K. S. Thorne *et al.* (Yale U. Press, New Haven CT, 1986).

Translated by D. Parsons