Quantum dissipation properties of a Josephson balance comparator

T. V. Filippov[1)]

Scientific-Research Institute of Nuclear Physics, M. V. Lomonosov Moscow State University, 119899 Moscow, Russia

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The quantum dissipation properties of a Josephson balance comparator, i.e., a system of two shunted junctions in a superconducting ring, are examined. The quantum-mechanical dynamics of a generalized coordinate of the system (the phase difference between the two junctions) is analyzed. Damping is taken into account in the Caldeira–Leggett model. The finite rate of change of the external magnetic flux is also taken into consideration. The temperature dependence of the parameter ΔI_x, which characterizes the effect of fluctuations on the system, is studied. Comparison of theoretical and experimental results leads to the conclusion that the changes in the parameter ΔI_x are of a quantum nature.

There is convincing experimental evidence for the existence of *secondary quantum* effects, such as a macroscopic quantum tunneling and a quantization of the energy spectrum[1] in Josephson-effect systems. Josephson systems are fundamentally *dissipative* systems. This circumstance limits and complicates efforts to experimentally observe secondary quantum effects, including a macroscopic quantum coherence.

It is accordingly worthwhile to analyze the dissipation properties of a specific Josephson system: the so-called balance comparator.[2] The temperature dependence of the probability for the switching of a comparator, stemming from internal fluctuations of the junctions making up the comparator, was studied experimentally in Ref. 3. In the present letter we show that the experimental results of Ref. 3 cannot be explained on the basis of the existing theory.[4] It becomes necessary to consider quantum fluctuations in a calculation of this switching probability. It is thus demonstrated theoretically and experimentally[3] that an apparatus with a current characteristic at the level of quantum fluctuations can be realized in practice.

Figure 1 shows a simplified schematic circuit of the comparator, which is formed by two Josephson junctions J_1 and J_2. Let us summarize the operating principle of this system. A current I_x is fed to the midpoint of the comparator, and the external magnetic flux Φ_e induces a circular current I_s in the circuit. If the total current I_s+I_x through one of the junctions exceeds the critical level, there will be a change in the Josephson phase difference at this junction, by an amount which is a multiple of 2π.

A plot of the switching probability of, say, junction J_2 versus the current I_x would be a vertical step if internal fluctuations in the junctions were ignored. When these fluctua-

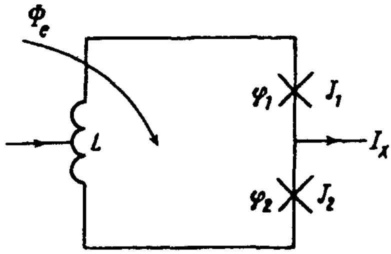

FIG. 1. Balance comparator: a system of two Josephson junctions in a superconducting ring.

tions are taken into account, the step is smeared out. As a measure of the extent of this smearing we can use the parameter

$$\Delta I_x = |dP_2/dI_x|^{-1}, \quad I_x = 0. \tag{1}$$

Our purpose in this letter is to calculate ΔI_x through an analysis of the quantum dynamics of a balance comparator within the framework of the Caldeira–Leggett model.[5]

We assume that each of the junctions in Fig. 1 has a critical current I_c and a capacitance C. We assume that each junction is shunted externally by a normal metal conductor with a resistance R. Under the condition

$$I_c R \ll \Delta(T)/e, \tag{2}$$

where $\Delta(T)$ is the energy gap of the superconductors used, the classical dynamics of such junctions can be described well by the resistive model of Ref. 6, while the quantum-mechanical properties can be described well by the Caldeira–Leggett model.[5] In the latter model the dissipation results from an interaction of a macroscopic variable of the system with an ensemble of harmonic oscillators forming a reservoir at equilibrium.

If the inductance of the ring, L, is sufficiently small ($\pi I_c L/\Phi_0 \ll 1$, where $\Phi_0 = h/2e$ is the quantum of magnetic flux), the relationship between the external magnetic flux $2\varphi_e(t) = 2\pi\Phi_e(t)/\Phi_0$ and the *resultant* phase on the comparator can be assumed to be linear.[4] We can thus assume that the resultant phase $2\varphi_e = \varphi_1 + \varphi_2$ is fixed, and we can restrict the analysis to the one-dimensional problem of the dynamics of the *difference* phase $2\varphi = \varphi_1 - \varphi_2$.

We write an expression for the potential energy of the system:[6]

$$U(\varphi)/2E_c = -\cos[\varphi_e(t)]\cos[\varphi] - i_x\varphi, \tag{3}$$

$$E_c = \Phi_0 I_c/2\pi, \quad i_x = I_x/2I_c.$$

It is easy to see that an increase in the phase $\varphi_e(t)$ over time leads to an inversion of the potential energy. Let us discuss the dynamics of this system at a qualitative level in the case of an instantaneous inversion of the energy. If the wave packet is initially localized at a minimum of the potential, then after the inversion the packet tends to move away from an unstable position near the maximum to the nearest local minimum. The direction of the motion is determined by the sign of i_x, in the absence of fluctuations. When there are fluctuations, there is a finite probability for the packet to change direction.

The effect of fluctuations is strongest near a maximum of the potential. We accordingly linearize (3) with respect to φ at $\varphi \approx 0$:

$$\tilde{U}(\varphi)/2E_c \simeq \mu(t)\varphi^2/2 - i_x\varphi. \tag{4}$$

We choose a time evolution $\cos[\varphi_e(t)]$ of the type

$$\cos[\varphi_e(t)] \equiv \mu(t) = (\mu_1+\mu_2)\exp(-2\kappa t) - \mu_2, \tag{5}$$

$$\mu_1 \equiv \cos[\varphi_e(0)], \quad \mu_2 \equiv |\cos[\varphi_e(\infty)]|,$$

where the parameter κ/ω_0 represents the rate of change of the potential energy of the system.

We are interested in the probability (P_2) for observing the system in the local minimum at the left after the inversion:

$$P_2 = \int_{-\infty}^{0} \rho(x,x,t)dx, \quad x \equiv \varphi. \tag{6}$$

This case corresponds to a switching of the lower junction, J_2, in Fig. 1. The quantity $\rho(x,x,t)$ in this expression is a diagonal element of the density matrix of the system.

At $t>0$, the density matrix, averaged over the degrees of freedom of the reservoir, is related to the initial matrix (at $t=0$) in a linear fashion in the $\eta-\xi$ coordinate representation[5] ($\eta = x+y$, $\xi = x-y$):

$$\rho(\eta,\xi,t) = (1/2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\eta_i d\xi_i \;\; J(\eta,\xi,t;\eta_i,\xi_i,0) \;\; \rho(\eta_i,\xi_i,0). \tag{7}$$

We choose the initial density matrix to be

$$\rho(\eta_i,\xi_i,0) = \frac{1}{(4\pi\sigma_{xi}^2)^{1/2}}\exp\left(-\frac{(\eta_i - 2x_i)^2}{16\sigma_{xi}^2} - \beta\gamma_Q^{-2}\sigma_{vi}^2\xi_i^2\right), \tag{8}$$

as the result of a Wigner transformation[7] from a classical Gibbs distribution with expressions for the coordinate variance and velocity variance generalized to the quantum-mechanical case:

$$\sigma_{xi}^2 = \frac{\gamma_Q}{2\pi\beta^{1/2}}\int_0^{\Omega/\omega_0} d\epsilon \, \coth\left(\frac{\hbar\omega_0}{2kT}\epsilon\right)\frac{\epsilon}{(\mu_1-\epsilon^2)^2+\beta^{-1}\epsilon^2}, \tag{9}$$

$$\sigma_{vi}^2 = \frac{\gamma_Q}{2\pi\beta^{3/2}}\int_0^{\Omega/\omega_0} d\epsilon \, \coth\left(\frac{\hbar\omega_0}{2kT}\epsilon\right)\frac{\epsilon^3}{(\mu_1-\epsilon^2)^2+\beta^{-1}\epsilon^2},$$

$$\beta = (\omega_c/\omega_0)^2, \quad \omega_c = 2\pi I_c R/\Phi_0, \quad \omega_0^2 = 2\pi I_c/\Phi_0 C,$$

$$\gamma_Q = \hbar\omega_0/2E_c, \quad x_i = i_x/\mu_1, \quad v = \omega_c^{-1}dx/dt.$$

Here Ω ($\gg \omega_0$) is the maximum frequency in the distribution of oscillators making up the reservoir,[5] T is the reservoir temperature, and k is the Boltzmann constant.

It is a simple matter to find the propagator of the system, $J(\eta,\xi,t;\eta_i,\xi_i,0)$, by the general scheme proposed by Caldeira and Leggett.[5] It turns out that a diagonal element of the density matrix at $t>0$ can be written in the simple form

$$\rho(x,x,t)=\frac{1}{(4\pi\sigma_x^2)^{1/2}}\exp-\left(\frac{(x-\langle x\rangle)^2}{4\sigma_x^2}\right), \tag{10}$$

where the mean coordinate $\langle x\rangle$ and the variance σ_x^2 are given by

$$\langle x\rangle=(x_iK+Q)/N, \tag{11}$$

$$\sigma_x^2=(\beta\gamma_Q^{-2}\sigma_{vi}^2+4\sigma_{xi}^2K^2+C)/4N^2, \tag{12}$$

$$K=(2\gamma_Q)^{-1}[\kappa/\omega_0\ \ z_0G^{-1}(z_0,z)dG(z_0,z)/dz_0+1/2\beta^{1/2}],$$

$$N=\sin(\nu\pi)/\pi\ \ \kappa/\omega_0\ \ \gamma_Q^{-1}G^{-1}(z_0,z)\ \ (z_0/z)^{\delta},$$

$$Q=1/2\ \ i_x\ \ \gamma_Q^{-1}\ \ \omega_0/\kappa\ \ G^{-1}(z_0,z)\ \ z_0^{\delta}\ \int_z^{z_0}G^{-1}(\theta,z)\theta^{-\delta-1}d\theta,$$

$$C=(2\pi\gamma_Q\beta^{1/2})^{-1}\int_0^{\Omega/\omega_0}d\epsilon\ \ \epsilon\coth\left(\frac{\hbar\omega_0}{2kT}\epsilon\right)C_\epsilon,$$

$$C_\epsilon=(\omega_0/\kappa)^2z_0^{2\delta}G^{-1}(z_0,z)\int_z^{z_0}\int_z^{z_0}d\theta_1d\theta_2G(\theta_1,z)G(\theta_2,z)W_\epsilon(\theta_1,\theta_2),$$

$$W_\epsilon(\theta_1,\theta_2)=\mathrm{Re}[\theta_1^{-\delta-i\omega_0\epsilon/\kappa-1}\theta_1^{-\delta+i\omega_0\epsilon/\kappa-1}],$$

$$G(\theta_1,\theta_2)=J_\nu(\theta_1)J_{-\nu}(\theta_2)-J_{-\nu}(\theta_1)J_\nu(\theta_2),$$

$$z=z_0\exp(-\kappa t),\ \ z_0^2=(\mu_1+\mu_2)/(\kappa/\omega_0)^2,\ \ \delta=1/2\beta^{1/2}(\kappa/\omega_0).$$

Here $J_\nu(z)$ and $J_{-\nu}(z)$ are the Bessel[8] functions of noninteger index

$$\nu=\frac{(1/4\beta+\mu_2)^{1/2}}{\kappa/\omega_0}. \tag{13}$$

It is easy to show that in the inverted potential the diagonal element of the density matrix $\rho(x,x,t)$ is a spreading packet moving in a direction determined by the sign of i_x. To show this situation, we write asymptotic expressions for the mean coordinate $\langle x\rangle$ and the variance σ_x^2 at times at which the coordinate of the particle reaches a value on the order of one:

$$\langle x\rangle=\pi/2\sin(\nu\pi)\ \ (z/z_0)^{\delta}\ \ J_{-\nu}(z)\ \ i_x\ \ (X_1+X_2), \tag{14}$$

$$X_1=\frac{z_0dJ_\nu(z_0)/dz_0+\delta J_\nu(z_0)}{(\kappa/\omega_0)^2(z_0^2+\delta^2-\nu^2)},\ \ X_2=z_0^{\delta}(\omega_0/\kappa)^2\int_0^{z_0}d\theta J_\nu(\theta)\theta^{-\delta-1},$$

$$\sigma_x^2=(\pi/2\sin(\nu\pi))^2\ \ (z/z_0)^{2\delta}\ \ J_{-\nu}^2(z)\ \ (\Sigma_1+\Sigma_2), \tag{15}$$

$$\Sigma_1=(\omega_0/\kappa)^2\beta\sigma_{vi}^2\ J_\nu^2(z_0)+\sigma_{xi}^2\ \ (\delta J_\nu(z_0)+z_0dJ_\nu(z_0)/dz_0)^2,$$

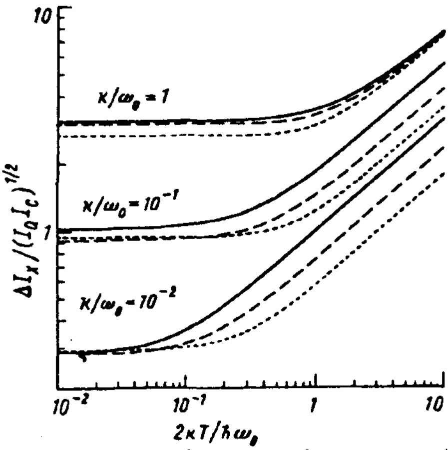

FIG. 2. Temperature dependence of the parameter ΔI_x for various values of the damping in the system. Solid curves—$\beta=1$; dashed curves—10; dotted curves—100. $\Omega/\omega_0=50$, $\mu_1=\mu_2=1$.

$$\Sigma_2=\frac{z_0^{2\delta}\gamma_Q}{2\pi(\kappa/\omega_0)^4\beta^{1/2}}\int_0^{\Omega/\omega_0}d\epsilon\ \epsilon\ \coth\left(\frac{\hbar\omega_0}{2kT}\ \epsilon\right)\int_0^{z_0}\int_0^{z_0}$$
$$\times d\theta_1 d\theta_2 J_\nu(\theta_1)\ J_\nu(\theta_2)W_\epsilon(\theta_1,\theta_2).$$

Using definition (6) of the probability P_2, for observing the system in the local minimum at the left after the inversion, we find the following expression for the parameter ΔI_x:

$$\Delta_{ix}\equiv\frac{\Delta I_x}{2I_c}=\left|\frac{(4\pi(\Sigma_1+\Sigma_2))^{1/2}}{X_1+X_2}\right|. \tag{16}$$

Figure 2 shows the temperature dependence of the parameter ΔI_x for various rates of change of the potential energy, κ/ω_0, and for various levels of the damping in the system, β. The parameter ΔI_x has been normalized to the quantity $(I_Q I_c)^{1/2}$, where the current $I_Q=(2\pi/\Phi_0)\hbar\omega_0/2$ characterizes the quantum fluctuations, by analogy with the magnitude of the "thermal current" [6] $I_T=(2\pi/\Phi_0)kT$.

In the high-temperature limit, $kT\gg\hbar\omega_0$ $(I_T\gg I_Q)$, the expression for the parameter ΔI_x is the same as that found previously[4] through a solution of the classical Fokker–Planck equation.

The temperature dependence of the parameter ΔI_x was studied in Ref. 3. Shunted Josephson junctions with the parameter values $I(4.2\ \text{K})=145\ \mu\text{A}$, $C=0.47$ pF, and $R=2$ Ω were used in the comparator circuit. The dimensionless inductance $\pi I_c L/\Phi_0$ was on the order of 0.1. A modeling of the equivalent circuit with the help of the PSCAN software package[9] showed that the rate of change of the potential, κ/ω_0, is between 0.05 and 0.15, and the parameters μ_1 and μ_2 are equal to 0.975.

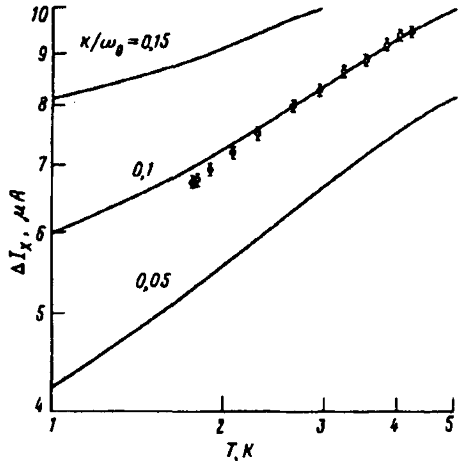

FIG. 3. Comparison of experimental data[2] on the temperature dependence of the parameter ΔI_x with theoretical predictions.

We need to stress that at the working temperatures the current I_Q is on the order of I_T $(\hbar\omega_0/2kT \approx 1)$. Accordingly, an analysis of the experimental data cannot be restricted to a consideration of the thermal noise. It is necessary to consider the contribution of quantum fluctuations to the comparator switching probability.

Figure 3 shows experimental points and theoretical curves for various values of the parameter κ/ω_0. Here we have made use of the Ambegoakar–Baratov expression[10] for the temperature dependence of the critical current of the junction (the transition temperature of the superconductors which were used was 8.8 K). There is a good agreement between theory and experiment with the value $\kappa/\omega_0 = 0.1$. Unfortunately, for this particular temperature range we cannot tell whether the parameter ΔI_x reaches the plateau characteristic of the quantum limit. The question requires further experimental study.

This theoretical analysis of the system makes it possible to plan several specific physics experiments. In addition to the temperature dependence of ΔI_x, with the formation of a plateau due to quantum fluctuations, it would be interesting to see how ΔI_x varies as a function of κ/ω_0, the rate of change of the external magnetic flux.

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1)e-mail: tfil@rsfq.npi.msu.su

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