Tricritical point in the Gross–Neveu model with a chemical potential and a nontrivial topology of the space

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The phase structure of the Gross–Neveu model is analyzed in a space with a topology $R^1 \times S^1$ and a chemical potential μ. In the $(\mu, 1/L)$ parameter plane, where L is the length of the S^1 circle, there exists a tricritical point, at which the boundaries of three different phases meet. © *1995 American Institute of Physics.*

We are still far from a unified theory of all forces in nature, including gravitational. This is the motivation for the substantial efforts of physicists to derive quantum field theories in spaces with nontrivial metrics and nontrivial topologies.[1] The relationship between spontaneous symmetry breaking and the curvature of the space, and also the topology of the space, has already been discussed along this approach.[2] In this letter we consider the joint effects of such factors as a nonzero number density of particles and a nontrivial topology of the space–time on the structure of the vacuum in quantum field theory.

We start from the 2D Gross-Neveu model,[3] whose Langrangian can be written

$$L=\sum_k \left(\bar{\psi}_k i\hat{\partial}\psi_k+\sigma\bar{\psi}_k\psi_k\right]-N\sigma^2/(2g), \tag{1}$$

where $\hat{\psi}_k$ is a two-component Dirac spinor for each fixed value $k=1,...,N$.[In the equations of motion we have $\sigma=\Sigma\bar{\psi}_k\psi_k$, and Eq. (1) is equivalent to the well-known Lagrangian with a four-fermion interaction.[3]] Lagrangian (1) is symmetric under the discrete chiral transformation $\sigma\to-\sigma$, $\psi_k\to\gamma^5\psi_k$. An attractive feature of this model is that many of its properties (asymptotic freedom, spontaneous breaking of chiral invariance, etc.) are reminiscent of quantum chromodynamics. This model can also be used to describe quasi-1D conducting compounds which have come to be known as "Peierls insulators."[4] This field theory was recently studied in a 2D space–time with the topology $R^1\times S^1$ (the spatial coordinate axis has been compactified here, and the S^1 circle has a length L). In contrast with Ref. 5, we consider the phase structure of the Gross–Neveu model on the space $R^1\times S^1$ as a function of two external parameters: the length of the circle, L, and the chemical potential $\mu\geq 0$.

We begin by recalling that the effective potential of the model with μ, $\lambda=0$ $(\lambda\equiv 1/L)$ takes the following form in the leading order of a $1/L$ expansion:[3,6]

$$V_0(\sigma)=\frac{N\sigma^2}{2g}+\frac{iN}{(2\pi)^2}\int d^2p\ln(\sigma^2-p^2)=\frac{N\sigma^2}{4\pi}[2\ln(\sigma/M)-1]. \tag{2}$$

In (2) and below we assume $\sigma\geq 0$. In addition, expression(2) reflects another property of the Gross–Neveu model: dimensional transmutation. In place of the dimensionless coupling constant g, the independent parameter of the model is the dimensional quantity M, whose physical meaning is the mass of the fermions.

With $\mu \neq 0$ and $\lambda=0$, the effective potential is[7]

$$V_\mu(\sigma)=V_0(\sigma)-\frac{N}{2\pi}\int_{-\infty}^{\infty}dp_1\Theta(\mu-\sqrt{\sigma^2+p_1^2})(\mu-\sqrt{\sigma^2+p_1^2}), \tag{3}$$

where $\Theta(x)=1$ at $x\geq 0$ and $\Theta(x)=0$ at $x<0$. Finally, with $\lambda \neq 0$ and $\mu=0$, and in the case in which the fields satisfy periodic boundary conditions, i.e., $\psi_k(x+L)=\psi_k(x)$, $\sigma(x+L)=\sigma(x)$, the effective potential is[5]

$$V_L(\sigma)=V_0(\sigma)-\frac{N}{\pi L}\int_{-\infty}^{\infty}dp_0\ln\left[1-\exp(-L\sqrt{\sigma^2+p_0^2})\right]. \tag{4}$$

In the present letter we restrict the analysis to periodic boundary conditions on the fields, although it is possible to study the Gross–Neveu model under antiperiodic conditions and also under conditions of a more general type.[5] The absolute minimum of the function (4) is at the point $\sigma_0(\lambda)$, which in the limit $\lambda\to 0$ is $\sigma_0(0)=M$, while in the limit $\lambda\to\infty$ it is

$$\sigma_0(\lambda)\sim\pi\lambda/\ln(\lambda/\lambda_0), \tag{5}$$

where $4\pi\lambda_0=M\exp(\gamma)$, and $\gamma=0.577...$ is Euler's constant.

We assume $\mu,\ \lambda \neq 0$, and we assume that the fields ψ and σ satisfy periodic boundary conditions. In order to find the effective potential in the leading order of a $1/N$ expansion in this case, all we need to do is replace the integration over the variable p_1 in Eqs. (2) and (3) by a summation over the discrete values $p_{1n}=2\pi\lambda n$ $(n=0,\pm 1,\pm 2,...)$. As a result, we find

$$V_{\mu L}(\sigma)=V_L(\sigma)-N\lambda\sum_{n=-\infty}^{\infty}\Theta(\mu-\sqrt{\sigma^2+(2\pi n\lambda)^2})(\mu-\sqrt{\sigma^2+(2\pi\lambda n)^2}). \tag{6}$$

We now restrict the discussion to value μ and λ, for which $(\mu,\lambda)\in\Omega$, where

$$\Omega=\{(\mu,\lambda):0\leq\mu\leq 2\pi\lambda\}. \tag{7}$$

In this case, only $V_L(\sigma)$ and the term of the sum corresponding to $n=0$ contribute to expression (6). The steady-state equation for function (6) is, by analogy with Ref. 5,

$$\frac{dV_{\mu L}(\sigma)}{d\sigma}=0=\frac{N\sigma}{\pi}\left\{\ln(\lambda/\lambda_0)+I(\sigma)-\frac{\pi\lambda}{\sigma}[\Theta(\mu-\sigma)-1]\right\}, \tag{8}$$

where

$$I(\sigma)=\sum_{n=1}^{\infty}\{1/n-[n^2+(\sigma/2\pi\lambda)^2]^{-1/2}\}. \tag{9}$$

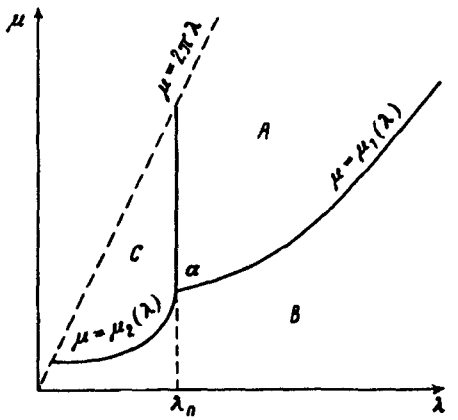

FIG. 1.

We thus see that under the condition $\lambda > \lambda_0$ the function in (6) has two stationary points: $\sigma_1 = 0$ and $\sigma_2 = \sigma_0(\lambda)$, where $\sigma_0(\lambda)$ is the point of the absolute minimum of function (4) [see (5)]. We consider the equation

$$V_{\mu L}(0) = V_{\mu L}(\sigma_0(\lambda)), \tag{10}$$

which specifies a critical curve $\mu = \mu_1(\lambda)$. Clearly, at $\mu > \mu_1(\lambda)$ and $\lambda > \lambda_0$ there are points corresponding to a massless, chirally invariant phase A. Here $\sigma_1 = 0$ is the absolute minimum of the potential $V_{\mu L}(\sigma)$. If $\mu < \mu_1(\lambda)$ and $\lambda > \lambda_0$, the absolute minimum of the potential is at $\sigma_2 = \sigma_0(\lambda)$. Corresponding to these points in the region Ω is massive phase B, in which the chiral invariance of the original model is spontaneously broken (Fig. 1). Since the order parameter [i.e., the position of the absolute minimum of the function $V_{\mu L}(\sigma)$] changes in value abruptly at the intersection of the curve $\mu = \mu_1(\lambda)$, this is a curve which represents first-order phase transitions. From (6)–(10) we find

$$\mu_1(\lambda) = [V_L(0) - V_L(\sigma_0(\lambda))]/(N\lambda). \tag{11}$$

It is thus a straightforward matter to show that in the limit $\lambda \to \infty$ we have

$$\mu_1(\lambda) \approx (\gamma + 1)\sigma_0(\lambda)/2 \approx \frac{(\gamma+1)\pi\lambda}{2\ln(\lambda/\lambda_0)}. \tag{12}$$

Furthermore, numerical calculations yield $\mu_1(\lambda_0) \approx (0.32...)2\pi\lambda_0$; i.e., the curve $\mu_1(\lambda)$ intersects the line $\lambda = \lambda_0$ at a point α (Fig. 1) which belongs to region Ω.

We now assume $\lambda < \lambda_0$. In this case, steady-state equation (8) has one more solution in addition to $\sigma_{1,2}$: $\sigma_3 = \tilde{\sigma}_0(\lambda)$. In the limit $\lambda \to \lambda_0$, this other solution can be written

$$\tilde{\sigma}_0(\lambda) \approx [8(\lambda_0 - \lambda)\lambda_0\pi^2/\zeta(3)]^{1/2}, \tag{13}$$

where $\zeta(3) = 1.202...$. At the point $\sigma_1 = 0$, the potential obviously has a local maximum. Consequently, the absolute minimum of this potential is at either the point $\tilde{\sigma}_0(\lambda)$ or $\sigma_0(\lambda)$, depending on whether we are above or below the curve

$$\mu_2(\lambda)=[V_L(\tilde{\sigma}_0(\lambda))-V_L(\sigma_0(\lambda))]/(N\lambda). \tag{14}$$

In the limit $\lambda\to\lambda_0$, the curve of $\mu_2(\lambda)$ touches the point α, at which we have

$$\mu_2(\lambda)\approx\mu_1(\lambda_0)-\tilde{\sigma}_0(\lambda). \tag{15}$$

When the curve $\mu_1(\lambda)$ is crossed, the coordinate of the absolute minimum of the potential changes in value abruptly. Accordingly, a first-order phase transition occurs from the massive phase B to the massive phase C of the theory. All the points in the Ω region which lie above $\mu_2(\lambda)$ and to the left of the straight line $\lambda=\lambda_0$ correspond to this new phase (Fig. 1). We should also point out that on the line $\lambda=\lambda_0$ above point α there is a second-order phase transition from phase C to phase A. Here $\tilde{\sigma}_0(\lambda)$ vanishes; i.e., the order parameter is continuous at the phase boundary.

In summary, we have shown that in the leading order of a $1/N$ expansion in the Gross–Neveu model in the space $R^1\times S^1$ with a chemical potential there exists a tricritical point α, at which three phases touch each other: two massive phases and one massless phase (Fig. 1). Three critical curves emerge from point α. Two of them are curves of first-order phase transitions. When these curves are intersected, there are abrupt changes in not only the masses of the fermions, but also the energy density of the ground state. The third curve represents second-order phase transitions.

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