Local modes and scattering of spin waves by a soliton in an isotropic 2D ferromagnet

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A Belavin–Polyakov soliton in an isotropic 2D ferromagnet with a topological charge ν has $2|\nu|$ local magnon modes of zero frequency. These modes are the limiting points of partial cylindrical waves with an azimuthal number $-|\nu|<m\leq|\nu|$ in the limit $k\to 0$. © *1995 American Institute of Physics.*

1. Solitons play a special role in the thermodynamics of 1D and 2D nonlinear models of ordered media, in particular, magnetic materials. Local modes, especially zero modes, are extremely important in the derivation of a soliton thermodynamics.[1-3] In the method of soliton phenomenology of Ref. 4, for example, local modes determine the temperature dependence of the density of 1D solitons. In addition, a resonance involving local modes can be observed directly in experiments.[5,6]

Several exact solutions describing either solitons or local modes against the background of solitons have been derived for 1D magnetic materials. For models of 2D magnetic materials which are of physical interest, there is only the Belavin–Polyakov exact solution,[7] which describes a π_2-topological soliton in a ferromagnet with an energy

$$W=A\int(\partial\mathbf{m}/\partial x_i)^2d^2x, \tag{1}$$

where A is an exchange constant, $\mathbf{m}=\mathbf{M}/M_0$ is a unit vector which determines the magnetization $\mathbf{M}$, and $M_0=|\mathbf{M}|$. There has been no study of the spectrum of magnons in the presence of a soliton.

In the present letter we show that the problem of the scattering of a spin wave by a soliton can be solved completely for the Landau–Lifshitz equation describing a ferromagnet with an energy as in (1). We analyze the local modes in this case.

2. To analyze small oscillations of the magnetization it is convenient to introduce a rotating system of unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, where $\mathbf{e}_3=e_z\cos\theta+\sin\theta\,(\mathbf{e}_x\cos\varphi+\mathbf{e}_y\sin\varphi)$ coincides with $\mathbf{m}$ in the soliton, $\mathbf{e}_2=-\mathbf{e}_x\sin\varphi+\mathbf{e}_y\cos\varphi$, and $\mathbf{e}_1=\mathbf{e}_2\times\mathbf{e}_3$. Here θ and φ are the angle variables for $\mathbf{m}$. Here is the explicit expression for the soliton solution:[7] $\tan(\theta/2)=(R/r)^{|\nu|}$, where $\varphi=\nu\chi+\varphi_0$, $\nu=\pm1,\pm2,\ldots$ is the topological charge; r, χ are polar coordinates in the plane of the ferromagnet; R is the radius of the soliton, an adjustable parameter; and φ_0 is another adjustable parameter. Linearizing the Landau–Lifshitz equation with respect to m_1 and m_2, we can write an equation for the spin waves against the background of a soliton in the form of a 2D Schrödinger equation for the quantity $\psi=m_1+im_2$:

$$\left(-\nabla^2+\frac{\nu^2}{r^2}\cos 2\theta\right)\psi-2i\cos\theta\,\frac{\nu}{r^2}\frac{\partial\psi}{\partial\chi}+i\,\frac{2A}{\gamma M_0}\frac{2\psi}{\partial t}=0, \qquad (2)$$

where γ is the gyromagnetic ratio. The solution of this equation is a superposition of cylindrical waves:

$$\psi=\sum_{m=-\infty}^{\infty}F_m(r)\exp(im\chi-i\omega t), \qquad (3)$$

where m is the azimuthal number, and the functions $F_m(r)$ satisfy the equations

$$-\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dF_m}{dr}\right)+\frac{1}{r^2}(m^2+2m\nu\cos\theta+\nu^2\cos 2\theta)F_m=k^2F_m,\quad k^2=\frac{\omega M_0}{2\gamma A}. \qquad (4)$$

The "potential" in this equation is not small, but it can be analyzed quite comprehensively in the long-wave limit $kR\ll 1$.

3. In the case $\omega=0$, we can write an exact solution of Eq. (4):

$$F_m^{(0)}=[\tan(\theta/2)]^{m\nu}\sin\theta=(R/r)^{\sigma m}\sin\theta,\quad \sigma=\nu/|\nu|. \qquad (5)$$

This exact solution exists because gauge invariance is restored in the limit $\omega\to 0$, and a self-duality equation holds. This equation is inherent in the static 2D Landau–Lifshitz equation (Ref. 7; see also Ref. 8). The functions in (5) can be found from the general solution of the self-duality equation, $(m_x+im_y)/(1-m_z)=w=f(z)$, $z=x+iy$, by choosing $f(z)=z^\nu+\alpha z^n$ and by taking the limit $\alpha\to 0$. The long-wave asymptotic behavior of the magnon wave functions against the background of more-complex soliton configurations, described by the solution $w=f(z)$, can be derived in the same way.

An elementary analysis shows that in the limit $r\to 0$ we have a solution $F_m^{(0)}(r)\propto(r/R)^{|\nu|-\sigma m}$, while in the limit $r\to\infty$ we have $F_m^{(0)}(r)\propto(r/R)^{-|\nu|-\sigma m}$. For values $-|\nu|<m\leq|\nu|$, the functions $F_m^{(0)}(r)$ thus describe magnon modes with a zero frequency which are localized at a soliton (for simplicity we assume $\nu>0$ below). For a soliton in a ferromagnet with the energy in (1), there are thus 2ν zero modes. The physical meaning of two of them is obvious: For $m=1$, we have a function $F_1\propto\theta_0'$, which describes translational modes, i.e., a displacement of the soliton as a whole. The case $m=0$ corresponds to changes in the adjustable parameters of the soliton, φ_0 and R. The other local modes with $\omega=0$, which may exist under the condition $\nu>1$, arise because of the high hidden symmetry of the static Landau–Lifshitz equation in model (1).

4. For small but nonzero values of the frequency ω, the solutions $F_m^{(0)}$ can be used as approximate solutions in the region $r\ll 1/k$, where the term ψk^2 in (4) is small in comparison with the terms containing $d^2\psi/dr^2$ or ψ/r^2. If r is instead large, $r\gg R$, we can use a different simplification: At $r\gg R$, we have $\theta\to 0$, and (4) becomes the standard Bessel equation. The solution of this equation is well known and can be written in the form

$$F_m=C_1J_n(kr)+C_2N_n(kr),\quad n=m+\nu, \qquad (6)$$

where $J_n(x)$ and $N_n(x)$ are Bessel and Neumann functions of integer index n (we will also be using the notation $n=m+\nu$ below). If $kR\ll 1$, there is a wide range of values of the variable r, namely $R \ll r \ll 1/k$, in which both approximate solutions, (5) and (6), are valid. Comparing the asymptotic behavior in (5) in the limit $r \rightarrow \infty$ and that in (6) for $kr \ll 1$, we easily see that these results are the same if the constants C_1 and C_2 are chosen in a certain way. It thus becomes possible to construct an asymptotically exact solution which becomes (5) at $r \sim R$ and which becomes (6) at $kr \gtrsim 1$. [This approach has been taken in an analysis of solitons of small radius $R \ll \Delta_0 = (A/K)^{1/2}$ in a ferromagnet with an anisotropy constant K (Ref. 9).] The asymptotic form of the solution $F_m(r)$ depends on m.

In the region in which local modes exist, $-\nu < m \leqslant \nu$, we need to choose $C_1 = 0$, $C_2 \neq 0$, and we find

$$F_m = -\sin\theta \left(\frac{r}{R}\right)^{\nu} \frac{\pi}{(n-1)!} \left(\frac{kR}{2}\right)^{n} N_n(kr), \quad -\nu < m \leqslant \nu. \tag{7}$$

Outside the region in which the local modes exist we need to choose $C_2 = 0$, $C_1 \neq 0$. For $m \leqslant -\nu$, we join $J_n(kr)$ with the solution $F_m^{(0)}$ in (5), which behaves well in the limit $r \rightarrow 0$. For $m > \nu$, we join it with the other solution of Eq. (4), which can be found easily from (5) by the standard method of variation of an arbitrary constant. As a result, we have

$$F_m = \sin\theta \left(\frac{r}{R}\right)^{\nu} (|n|!) J_{|n|}(kr), \quad m \leqslant -\nu, \tag{8}$$

$$F_m = \sin\theta \left(\frac{r}{R}\right)^{\nu} \left[\frac{1}{n} + \frac{2}{m}\tan^2\frac{\theta}{2} + \frac{1}{m-\nu}\tan^4\frac{\theta}{2}\right] n! \left(\frac{2}{kR}\right)^{n} J_n(kr), \quad m > \nu. \tag{9}$$

5. Using the explicit expression for $F_m(r)$ [Eqs. (7)–(9)], we can construct a general solution for small oscillations of the magnetization. It is more convenient to write the solution in terms of the variable $\tilde{\psi} = \psi \exp(i\nu\chi - i\omega t)$, which in the limit $r \rightarrow \infty$ becomes $(m_x + im_y)\exp(-i\omega t)$ and describes a spin wave against the background of a uniform state $\mathbf{m} \| \mathbf{e}_z$. In general, the asymptotic behavior of $\tilde{\psi}$ in the limit $r \rightarrow \infty$ can be written as follows, where we are using (5):

$$\tilde{\psi} = \sum_{n=-\infty}^{+\infty} c_n e^{in\chi} [e^{-ikr} + S_n(k) e^{ikr}]/r^{1/2}. \tag{10}$$

Here c_n are arbitrary complex constants, and $S_n(k) = \exp[2i\delta_n(k)]$ are elements of the scattering matrix. [The plus sign in square brackets arises here, in contrast with the standard situation for a 3D soliton, because of the difference between the asymptotic expressions for free motion: $(1/r)\sin kr$ for 3D and $J(kr) \propto (1/r^{1/2})\cos kr$ for 2D.] The form of $S_m(k)$ at $kR \ll 1$ depends strongly on m: In the region in which local modes exist, $-\nu < m \leqslant \nu$, we have $S_m(k) = -1$, and the scattering is at its strongest. Outside the region of local modes we have $S_m(k) = 1$. In other words, the corresponding partial waves behave as asymptotically free waves.

By choosing the constants c_n in a certain way, we can write a general solution of the problem of the scattering of a plane spin wave by a 2D soliton:

$$\tilde{\psi} = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\chi) e^{ikr}/\sqrt{r}, \quad f(\chi) = -(2/\pi k)^{1/2} e^{-i\pi/4} \sum_{m=-\nu+1}^{\nu} e^{i(m-\nu)\chi}. \tag{11}$$

As we have already mentioned, only the partial waves in the region in which local modes exist contribute to the scattering function $f(\chi)$. This function is highly anisotropic, vanishing for $2\nu-1$ values of χ.

The approach proposed above (constructing exact quasistatic solutions on the basis of the duality equations at $r \ll 1/k$ and joining them with known solutions which describe the dynamics of spins far from a soliton) can also be taken for other 2D problems. For example, the analysis of local modes for the Lorentz–invariant σ model, which describes an isotropic classical antiferromagnet, would be carried out in essentially the same way, and it would lead to similar results (all we have to do is change in the relationship between ω and k to $\omega = ck$, where c is the phase velocity of the free magnons). As was mentioned above, this approach can also be taken in the case of more-complex soliton configurations, and also to analyze the dynamics of a soliton in a magnetic material with a finite area. (The latter problem is important for an analytic description of the results of numerical simulations of the motion of solitons, which are always carried out for a system of finite dimensions.) On the other hand, generalizing this approach to the case of anisotropic magnetic materials is not a trivial problem.

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[1] Yu. A. Izyumov, Usp. Fiz. Nauk **155**, 553 (1988) [Sov. Phys. Usp. **31**, 689 (1988)].
[2] R. Rajaraman, *An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1982).
[3] V. G. Bar'yakhtar and B. A. Ivanov, Sov. Sci. Rev. A. Phys. **16**, No. 3, 1 (1992), ed. by I. M. Khalatnikov.
[4] J. R. Currie, J. A. Krumhansl, A. R. Bishop, and S. E. Trullinger, Phys. Rev. B **22**, 477 (1980).
[5] G. A. Kraftmakher, V. V. Meriakri, A. Ya. Chervonenkis, and V. I. Sheglov, Zh. Éksp. Teor. Fiz. **63**, 1353 (1972) [Sov. Phys. JETP **36**, 714 (1972)].
[6] J. P. Boucher, G. Rius, and Y. Henre, Europhys. Lett. **4**, 1073 (1987).
[7] A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975).
[8] A. M. Perelomov, Usp. Fiz. Nauk **134**, 577 (1981) [Sov. Phys. Usp. **24**, 645 (1981)].
[9] V. P. Voronov, B. A. Ivanov, and A. M. Kosevich, Zh. Éksp. Teor. Fiz. **84**, 2235 (1983) [Sov. Phys. JETP **57**, 1303 (1983)].

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