Orbital angular momentum of vortices and textures due to spectral flow through the gap nodes: example of the ^{3}He-A continuous vortex

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The orbital angular momentum of the axisymmetric textures and vortices in Fermi superfluids and superconductors is discussed using the example of ^{3}He-A. If there are no zeros in the quasiparticle spectrum, the orbital momentum of the texture is robust, i.e, it is not sensitive to the change of the texture provided that the axial symmetry is not violated. If zeros exist or there is an anomalous branch of the low-energy fermions in the vortex core, the orbital angular momentum will depend on the texture. This dependence comes from the accumulation of the fermionic topological charge induced by the texture. The change of the orbital angular momentum in the texture occurs as a result of spectral flow through the nodes or along the anomalous branch. © *1995 American Institute of Physics.*

1. INTRODUCTION. AXIAL SYMMETRY CHARGE

Let us consider the static orbital angular momentum of the axisymmetric distribution of the order parameter in the Fermi superfluids and superconductors. We apply the general approach of the spectral flow, which is valid for the quantized vortices in conventional superconductors and ^{3}He-B, where the vortices have singular cores, and for the continuous order parameter texture which can exist in ^{3}He-A. The general result, when applied to the ^{3}He-A texture, allows us to treat the angular momentum paradox in ^{3}He-A.

The paradox of the orbital angular momentum in the Fermi pair-correlated states has a long history, which began in 1961 when Anderson and Morel[1] introduced the anisotropic state which later was found to be the A-phase of superfluid ^{3}He. Each Cooper pair in this state has an orbital angular momentum $\hbar\mathbf{l}$, where $\mathbf{l}$ is the unit vector of the orbital anisotropy. Estimates of the total angular momentum of the system vary from $(N/2)\hbar\mathbf{l}$, which corresponds to the momentum $\hbar/2$ per N particles of the system (see Ref. 2), to a much smaller quantity $\sim N(\Delta_0/E_F)^2\hbar\mathbf{l}$ (see a recent paper[3]; here Δ_0 is the gap amplitude which is much smaller than the Fermi energy E_F). The latter estimate corresponds to the space integral of the intrinsic dynamic momentum found by Cross,[4] which is related to the inertia of the $\mathbf{l}$ precession.

We show here that this paradox is closely related to the axial anomaly which appears either due to zeros in the quasiparticle spectrum[5–7] or due to quantized vortices.[8] In each case the spectral flow through the gap nodes or along the anomalous branch of fermions within the vortex core leads to the accumulation of the orbital angular momentum. The latter plays the part of the topological charge induced in the Fermi sea by the texture of the order parameter (the fermionic topological charge formalism in superfluid ^{3}He is discussed in Ref. 6).

The relevant fermionic charge in the axisymmetric system is related to the residual symmetry of the system. This is the generalized angular momentum, expressed in terms of the angular momentum and the particle number operators (see, e.g., review[9]):

$$\mathbf{Q}=\mathbf{L}_z-(n/2)\mathbf{N}. \tag{1.1}$$

Here n is an integer: $n=1$ for the homogeneous A-phase with $\mathbf{l}=\hat{z}$, where the rotational symmetry $SO(3)_L$ is spontaneously broken together with the gauge symmetry $U(1)_N$, but the combined symmetry with the generator $\mathbf{Q}=\mathbf{L}_z-(1/2)\mathbf{N}$ is conserved. This means that the action of $\mathbf{Q}$ on the (multicomponent) order parameter Ψ annihilates the order parameter: $\mathbf{Q}\Psi=0$; i.e., the A-phase order parameter does not change if the rotation is accompanied by the proper gauge transformation which is generated by $\mathbf{N}$ and which leads to the change of the order-parameter phase.

Equation (1.1) can be also applied to the inhomogeneous vacuum,[9] e.g., to the quantized vortices in the conventional s-wave, pair-correlated state: In this case n is the winding number of the vortex. In the inhomogeneous case the generator $\mathbf{L}_z$ contains two terms: $\mathbf{L}_z=\mathbf{L}_z^{\text{internal}}+\mathbf{L}_z^{\text{external}}$, where $\mathbf{L}^{\text{internal}}$ is the generator of the orbital rotations in the order parameter space [in the isotropic $SO(3)_L$ group], while $\mathbf{L}^{\text{external}}=i\mathbf{r}\times\partial_{\mathbf{r}}$ is the generator of the coordinate $\mathbf{r}$ rotations. The axisymmetric or $\mathbf{Q}$-symmetric state means that $\mathbf{Q}\Psi(\mathbf{r})=0$.

This symmetry of the vacuum tells us that Q is the conserved, integer (or half-odd-integer) quantum number, but this fact does not mean that the Q charge of the vacuum should be exactly zero. In the pure fermionic description (Sec. 4) the total charge of the vacuum is

$$\langle \text{vac}|\mathbf{Q}|\text{vac}\rangle=\sum_{Q,p_z,s} Q\theta(-E_{Q,p_z,s}), \tag{1.2}$$

where $E_{Q,p_z,s}$ are the energy eigenvalues of the fermions in the axisymmetric field of the order parameter (in addition to the quantum number Q, there are the other quantum numbers: the linear momentum p_z along the symmetry axis z, the radial quantum number s, etc.); $\theta(-E_{Q,p_z,s})$ is the step function of the energy, which shows that only the negative energy states contribute to the vacuum charge. The charge of the vacuum can be nonzero if some discrete symmetry is broken and $E_{Q,p_z,s}\neq E_{-Q,\pm p_z,s}$.

One can find the condition when this charge is zero, which for the A-phase state means that the total angular momentum $L_z=\langle\text{vac}|\mathbf{L}_z|\text{vac}\rangle=(n/2)N$, in accordance with Ref. 2. This condition is related to the adiabatic process, which means that during the process there is no level flow from or into the vacuum state, and thus the fermionic charge is conserved in this process. This process takes place, for example, if there is a gap in the fermionic spectrum. For the A-phase state it occurs in the limiting case of the Bose condensate of the isolated Cooper molecules each with a momentum $\hbar$ (or in a thin film in which the gap nodes disappear due to the transverse quantization; see Chap. 9 in Ref. 7. Let us start from the Bose condensate as the initial state which has $Q=0$ (and thus

$L_z=N/2$), and transform this state adiabatically into the real ^{3}He-A, without violation of the axisymmetry: In the final ^{3}He-A state we will then also have $Q=0$. The problem, however, is that during this continuous transformation the gap nodes appear at a certain time and the process can lose the adiabaticity, since the spectral flow through the nodes can emerge in the bulk liquid or at the boundary of the system.

Let us now determine how the spectral flow leads to a nonzero Q in the vacuum state of the axisymmetric texture in the arbitrary pair-correlated system and apply the result to the continuous vortex in the A-phase. This is a simple quantized vortex, in which the microscopic calculations (Sec. 4) can be completed and compared to the phenomenological hydrodynamic approach (Sec. 3). But let us first recall how the spectral flow modifies the linear momentum in the A-phase.

2. LINEAR MOMENTUM ANOMALY IN ^{3}He-A

Let us start from the Bose condensate of the isolated Cooper molecules with the symmetry of the A-phase or from the A-phase state with a negative chemical potential $\mu<0$, whose fermionic spectrum $E=\sqrt{(p^2/2m_3-\mu)^2+c^2(\mathbf{l}\times\mathbf{p})^2}$ has no nodes. One can adiabatically transform these two states into each other and therefore they have identical properties. The mass current (or the density of the linear momentum) in these node-free superfluids at $T=0$ is

$$\mathbf{j}_{\text{node-free}}=\frac{\hbar}{2m_3}\,\rho\mathbf{v}_s+\frac{1}{2}\,\nabla\times\mathbf{L}_{\text{node-free}}. \tag{2.1}$$

The first term is dictated by the Galilean invariance. Here $\mathbf{v}_s$ is the superfluid velocity in units of $\hbar/2m_3$. The vector $\mathbf{L}$ is the density of the angular momentum, which for the node-free states is

$$\mathbf{L}_{\text{node-free}}=\frac{\hbar}{2m_3}\,\rho\mathbf{l}. \tag{2.2}$$

Let us now continuously transform the node-free liquid into the real A-phase by changing the chemical potential from negative to positive. At $\mu>0$ the gap nodes appear at two points $\mathbf{p}=\pm p_F\mathbf{l}$, where $p_F^2/2m_3=\mu$. Near each node the fermions can be described as chiral Weyl fermions moving in an "electromagnetic" field $\mathbf{A}=p_F\mathbf{l}$ produced by the $\mathbf{l}$ texture.[7] If $[\partial_t\mathbf{A}\cdot(\nabla\times\mathbf{A})]\neq 0$ there is an effect of axial anomaly:[10] The spectral flow of fermions leads to a creation of quasiparticles from the vacuum. In ^{3}He-A each created quasiparticle carries the linear momentum $p_F\mathbf{l}$. This results in the production of the net quasiparticle linear momentum:

$$\partial_t\mathbf{P}_{qp}=\frac{1}{2\pi^2}\int d^3r\; p_F\mathbf{l}\;\;[\partial_t\mathbf{A}\cdot(\nabla\times\mathbf{A})]. \tag{2.3}$$

Since the total linear momentum is conserved, the momentum is transferred from the collective variables of the order parameter to a system of quasiparticles.

We take an arbitrary but fixed $\mathbf{l}(\mathbf{r})$ texture in the node-free state and consider the transformation into the real A-phase in such a way that only μ changes with time. At some instant $t=t_0$ the Fermi momentum appears. This momentum then changes from

$p_F=0$ at $t=t_0$ to the equilibrium value $p_F(\infty)$ in the real A-phase at $t=\infty$. In this process $\partial_t\mathbf{A}=\mathbf{l}\partial_t p_F$ and the total momentum of the texture, compared with Eq. (2.1), changes as follows:

$$\mathbf{P}(\infty)-\mathbf{P}(t_0)=-\int_{t_0}^{\infty}dt\ \partial_t\mathbf{P}_{qp}=-\frac{1}{2\pi^2}\int_{t_0}^{\infty}dt\int d^3r\ p_F^2\partial_t p_F\mathbf{l}\ [\mathbf{l}\cdot(\nabla\times\mathbf{l})]$$

$$=-\frac{1}{2}\int d^3r\ C_0\mathbf{l}\ [\mathbf{l}\cdot(\nabla\times\mathbf{l})],\quad C_0=\frac{1}{3\pi^2}\,p_F^3(\infty), \tag{2.4}$$

where $\mathbf{P}(t_0)=\int d^3r\ \mathbf{j}_{\text{node-free}}$ is the anomaly-free momentum in Eq. (2.1). The extra mass current in the A-phase,

$$\mathbf{j}_{\text{anomalous}}=-\frac{1}{2}\,C_0\mathbf{l}[\mathbf{l}\cdot(\nabla\times\mathbf{l})], \tag{2.5}$$

results from the helicity of the A field (concerning the role of helicity in particle physics, see Ref. 11).

3. ANGULAR MOMENTUM OF THE ^{3}He-A TEXTURE: A PHENOMENOLOGICAL APPROACH

Let us estimate the Q-charge of the vacuum in the A-phase for different continuous axisymmetric l-textures, which can be obtained by continuous deformation of the homogeneous vacuum with $\mathbf{l}=\hat{z}$.

The general solution of the axisymmetry equation $\mathbf{Ql}(\mathbf{r})=0$ for the $\mathbf{l}$ texture is

$$\mathbf{l}=\hat{z}\cos\eta(r)+\sin\eta(r)[\hat{r}\cos\alpha(r)+\hat{\phi}\sin\alpha(r)]. \tag{3.1}$$

We set $\eta(0)=0$ to have $\mathbf{l}(r=0)=\hat{z}$ at the center of the vessel. This is required by the continuity of the deformation of the homogeneous state with $\mathbf{l}=\hat{z}$.

If $\eta(r=r_0)=\pi$, the texture represents the continuous Anderson–Toulouse–Chechetkin 4π vortex in ^{3}He-A (Ref. 12) with

$$n=\frac{1}{2\pi}\oint_{r>r_0}d\mathbf{r}\cdot\mathbf{v}_s=\frac{1}{2\pi}\int d\mathbf{S}\cdot\nabla\times\mathbf{v}_s=\frac{1}{2\pi}\int dx\ dy\ \mathbf{l}\cdot(\partial_x\mathbf{l}\times\partial_y\mathbf{l})=2, \tag{3.2}$$

and with the core radius r_0. Here we used the Mermin–Ho relation[13]

$$\nabla\times\mathbf{v}_s=\tfrac{1}{2}\,e_{ijk}l_i\nabla l_j\times\nabla l_k \tag{3.3}$$

and the expression for the topological invariant which describes the mapping $S^2\to S^2$ of the vortex cross section to the sphere S^2 of the unit vector $\mathbf{l}\cdot\mathbf{l}=1$. The invariant shows that within the continuous 4π vortex the whole area 4π of the sphere is swept once. For simplicity we consider the coordinate to be independent of α.

In the phenomenological description the orbital angular momentum is given by the momentum of the current: $\mathbf{L}=\int d^3r\ \mathbf{r}\times\mathbf{j}$. The integration of the regular terms, [Eq. (2.1)] after integration by part and after using the boundary condition $\rho(R)=0$ (there are no particles outside the vessel of radius R), gives the standard contribution to the angular momentum: $L_z(\text{regular})=\frac{1}{2}N$. Thus the charge Q of the axisymmetric vacuum is given by the orbital momentum of the anomalous current:

$$\langle \text{vac}|\mathbf{Q}|\text{vac}\rangle=\int d^3r\ \hat{z}\cdot(\mathbf{r}\times\mathbf{j}_{\text{anomalous}})=-\frac{1}{2}\int d^3rC_0\ [\hat{z}\cdot(\mathbf{r}\times\mathbf{l})]\ [\mathbf{l}\cdot(\mathbf{\nabla}\times\mathbf{l})]$$

$$=-\pi L\int_0^{r_0} dr\ r^2C_0\ \sin^2\alpha\ \sin\eta\left(\partial_r\eta+\frac{\sin\eta\ \cos\eta}{r}\right), \tag{3.4}$$

(here L is the length of the vortex). This means that if the $\mathbf{l}$ texture contains a helix (i.e., if $\sin\alpha\neq 0$), the total momentum of the vortex texture in the A-phase will be reduced compared with $(1/2)N$ which was calculated for the node-free models.

In the following section this phenomenological expression is rederived from the general expression obtained by using the spectral flow along the anomalous branch of the fermions localized in the vortex core.

4. ORBITAL ANGULAR MOMENTUM FROM FERMION ZERO MODES ON VORTICES

In particle physics the fermion zero modes on strings are the p_z modes, i.e., they correspond to the branches of the spectrum of fermions localized in strings, $E_{Q,p_z,s}$, which cross zero energy as a function of the continuous parameter p_z. For the condensed matter strings, vortices in the pair-correlated systems, the important zero modes are Q modes,[14] the branches of spectrum $E_{Q,p_z,s=0}$, which cross zero energy as a function of the parameter Q. In most cases the charge Q can be considered as continuous. The Q zero modes in the condensed matter have the property of the p_z modes in particle physics: the algebraic sum of zero modes is nonzero and is defined by the winding number n of the vortex.[8] This means that the number ν of the negative fermionic levels with a given p_z is different for large positive and large negative Q:

$$\nu(p_z,Q=+\infty)-\nu(p_z,Q=-\infty)=2n, \tag{4.1}$$

and the branch $E_{Q,p_z,s=0}$ crosses zero at some $Q=Q_0(p_z)$.

In the most symmetric vortices $Q_0(p_z)=0$, i.e., the energy spectrum of the fermions on the anomalous branch, $E_{Q,p_z,s=0}=Q\omega(p_z)$, crosses zero at $Q=0$. However, if some discrete symmetry is broken in the vortex core, then we have $Q_0(p_z)\neq 0$. Such a situation was found in the continuous vortices in the A-phase if the helicity of vector $\mathbf{l}$ is nonzero.[15] According to Kopnin,[15,16]

$$Q_0(p_z)=r(p_z)\ \sin\alpha\ \sqrt{p_F^2-p_z^2}, \tag{4.2}$$

where $r(p_z)$ is the radius at which

$$\cos\eta(r)=\frac{p_z}{p_F}, \tag{4.3}$$

and the lowest energy levels with the radial quantum number $s=0$ are given by

$$E(Q,p_z,s=0)=\frac{\Delta_0}{p_F\ r(p_z)\ \cos\alpha}[Q-Q_0(p_z)]. \tag{4.4}$$

Although Q is discrete, the distance between the Q levels $\Delta_0/(p_F\ r(p_z)\cos\alpha)$ is very small compared with the gap amplitude Δ_0. This means that the effective Q is large and that it can be considered to be continuous.

Since $Q_0(p_z)$ depends on $\sin\alpha$ during the evolution of the vortex structure, the Q levels cross the zero energy, which leads to an accumulation of the charge Q in the vacuum. This situation was phenomenologically discussed in Sec. 3. In terms of the fermionic levels, the rate of the charge Q production,

$$\partial_t\langle\mathrm{vac}|\mathbf{Q}|\mathrm{vac}\rangle=\sum_{Q,p_z} Q\partial_t\nu(Q,p_z), \tag{4.5}$$

can be found from the following consideration. If $Q_0(p_z)$ changed due to the modification of the vortex, e.g., due to the change of α, the rate of the flow of the Q levels through zero will be $\partial_t Q_0(p_z)$. Since at each event the charge $Q_0(p_z)$ is transferred from the vacuum to the fermionic degrees of freedom, the total rate of the charge transfer is

$$\partial_t\langle\mathrm{vac}|\mathbf{Q}|\mathrm{vac}\rangle=\sum_{p_z} Q_0(p_z)\partial_t Q_0(p_z). \tag{4.6}$$

Thus if we start from the most symmetric vortex and continuously transfer this state into the vortex with a broken symmetry, we obtain the following general result for the charge Q of the vortex:

$$\langle\mathrm{vac}|\mathbf{Q}|\mathrm{vac}\rangle=\frac{1}{2}\sum_{p_z} Q_0^2(p_z). \tag{4.7}$$

Now we can apply this general result to the A-phase vortex. Using Eq. (4.2), we obtain the Q charge of the helical texture

$$\langle\mathrm{vac}|\mathbf{Q}|\mathrm{vac}\rangle=\sin^2\alpha\ \ L\int\frac{dp_z}{2\pi}\ \ r^2(p_z)\ (p_F^2-p_z^2). \tag{4.8}$$

We can show that this is in fact Eq. (3.4). According to Eq. (4.2), the function $r(p_z)$ is the inverse function of $p_z(r)=p_F(r)\cos\eta(r)$. Adding to Eq. (4.7) the factor $1=\int_0^R dr\,\delta[r-r(p_z)]$ (where R is the radius of the vessel), we obtain

$$\begin{aligned}\int\frac{dp_z}{2\pi}r^2(p_z)(p_F^2-p_z^2)&=\int\frac{dp_z}{2\pi}\int_0^R dr\,\delta[r-r(p_z)]\quad r^2(p_z)\ (p_F^2-p_z^2)\\&=\frac{1}{2\pi}\int_0^R dr\ r^2p_F^2(r)\ \sin^2\eta\ \ \partial_r(p_F\cos\eta)\\&=\frac{1}{2\pi}\int_0^R dr\bigg[r^2p_F^3(r)\sin^2\eta\ \ \partial_r\cos\eta\\&\qquad+r^2\sin^2\eta\cos\eta\partial_r\frac{p_F^3}{3}\bigg].\end{aligned} \tag{4.9}$$

The second term is integrated by parts using the condition $p_F=0$ outside the vessel, and we have

$$\int \frac{dp_z}{2\pi} r^2(p_z)\,(p_F^2-p_z^2)=-\frac{1}{3\pi}\int_0^{r_0} dr\; r^2 p_F^3 \sin\eta \left(\partial_r \eta+\frac{\sin\eta+\cos\eta}{r}\right), \tag{4.10}$$

which corresponds to Eq. (3.4).

5. CONCLUSION

The orbital angular momentum of the axisymmetric vacuum in the pair-correlated fermionic system is $L_z=(n/2)N+Q$, where N is the number of particles, n is an integer, and Q is the conserved fermionic charge in the axisymmetric vacuum. In the presence of the gap nodes the charge Q, given by Eq. (4.7), depends on the texture. The gap nodes, which give rise to $Q\neq 0$, can exist (i) in the bulk liquid (like in the A-phase), (ii) within the core of vortices, and (iii) on the surface of the container (the effect of the surface will be discussed later). When the texture changes, the charge Q is accumulated by the flow of the Q levels through zeros. This occurs only if a discrete symmetry is violated in the texture; in ^{3}He-A the effect exists only in the presence of the helical structure, with $\mathbf{l}\cdot(\nabla\times\mathbf{l})\neq 0$.

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