Shape factor in the critical state of superconductors

Yu. E. Kuzovlev

Physicotechnical Institute, National Academy of Sciences, Ukraine 340114 Donetsk, Ukraine

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An approach is proposed for describing on the basis of Bean's model the evolution of the critical state of type-II superconductors taking into account the real geometric characteristics of the sample. © *1995 American Institute of Physics.*

1. A number of properties of type-II superconductors which exhibit flux line pinning, as we know, can be described on the basis of the concept of a quasiequilibrium "critical state" characterized by the balance of the Lorentz force and the friction forces resulting from the pinning of the flux lines. In the simplest case (Bean's model) the maximum (critical) friction force at rest $\sim \Phi_0 j_c$, which acts on a unit length of a flux line and which determines the critical current density j_c, does not depend on the magnetic induction, i.e., on the concentration or direction of the flux lines. Even in this case, however, to extract the critical current density from diamagnetic measurements, it is necessary to know the geometric characteristics of the critical state. For example, the diamagnetic response in the case of the magnetization of a sample frozen in zero field is determined, aside from j_c, by the evolution of the shape of the surface bounding a region where the magnetic induction remains equal to zero. Specifically, it is of interest to determine the external field H_m in which this region vanishes and the diamagnetic moment of the sample saturates.

These characteristics are known only for the limiting cases of a long cylinder oriented along the field[1] and a very thin disk[2–4] (or a thin strip[5]) oriented perpendicular to the field. In the present paper we shall consider a more realistic geometry in which all dimensions are finite. It is shown that an expansion in spherical harmonics makes it possible to obtain an explicit expression for H_m as well as a chain of "integrals of motion" (functionals of the shape of the boundary of the screened region which do not change when the external field changes) that could be helpful in numerical modeling. For simplicity, we assume that the sample has a convex surface and axial symmetry, and that its axis is parallel to the external field. We also assume that the fields which are important for the critical state are much greater than H_{c1} and that the superconductor is isotropic in a plane perpendicular to the external field.

2. For a sufficiently weak external field H_e the sample contains a screened region where the total induction and the current are zero. Because of the convexity of the surface of the sample, it can be assumed that this region is simply connected and that its boundary is also convex. We place the origin of a polar coordinate system on the rotational symmetry axis Z inside this region. The current and the vector potential can then be written in the form

$$j_x = -J(r,u)\sin\varphi, \quad j_y = J(r,u)\cos\varphi,$$

$$A_x = -F(r,u)\sin\varphi, \quad A_y = F(r,u)\cos\varphi,$$

where $u \equiv \cos\vartheta$, $J(r,u) = -j_c$ for $R(u,H_e) < r < R_0(u)$ and $J(r,u) = 0$ in the opposite case, and $r = R_0(u)$ and $r = R(u,H_e)$ are, respectively, the equation of the surface of the sample and the boundary of the screened region. For the interior of the screened region, i.e., for $r < R(u,H_e)$, using the standard expansion in spherical harmonics, we easily obtain

$$F(r,u) = \tfrac{1}{2}H_e r(1-u^2)^{1/2} - \sum_{n\geq 0} r^{n+1} Q_n(u) I_n(H_e), \tag{1}$$

where

$$I_n(H_e) = \frac{j_c}{(2n+3)(n-1)} \int [R^{1-n}(u,H_e) - R_0^{1-n}(u)] Q_n(u) du \tag{2}$$

and

$$Q_n(u) \equiv Y_{n+1,1}(u) = (1-u^2)^{1/2} P_n^{(1,1)}(u),$$

$P_n^{(1,1)}(u)$ are the Jacobi polynomials, $Y_{LM}(u)$ are the spherical harmonics with the argument $u = \cos\vartheta$ which are normalized to 1, and ϑ is the polar angle. For $n = 1$ the integral $I_1(H_e)$ can be clearly expressed in terms of the logarithm $\ln(R_0/R)$. The first term on the right-hand side of the expression (1) refers to the external field and the sum refers to the field produced by the supercurrent.

Since the external field is completely cancelled out, the coefficients for all powers of r in the expansion (1) must vanish. Since $Q_0(u) = 1/2\sqrt{3(1-u^2)}$, it therefore follows that

$$\int [R_0(u) - R(u,H_e)] \sqrt{1-u^2} du = 2H_e/j_c, \tag{3}$$

$$I_n(H_e) = 0 \quad (n \geq 1). \tag{4}$$

The equality (3) gives an explicit rule for determining the characteristic field H_m. Specifically, we place the origin of coordinates at the point to which the region free of magnetic flux contracts. For $H_e = H_m$, we then have $R(u,H_e) = 0$ and therefore

$$H_m = j_c \int R_0(u) \sqrt{1-u^2} du/2. \tag{5}$$

Assuming axial symmetry, this limit point lies on the rotation axis Z and its position on the Z axis is evidently determined by the condition that the integral in (5) is maximum. If, however, the sample has an additional reflection symmetry relative to a plane perpendicular to the field, then the limit point lies at the center of the sample. For a sample in the shape of an ellipsoid of revolution, with the horizontal and vertical semiaxes in the ratio $a{:}b$, we thus find

$$\begin{aligned} H_m^{\text{ell}} &= j_c b[K(k) - E(k)]/k^2, \quad k \equiv (1-b^2/a^2)^{1/2}, \quad a > b; \\ H_m^{\text{ell}} &= j_c a[E(k) - (1-k^2)K(k)]/k^2, \quad k \equiv (1-a^2/b^2)^{1/2}, \quad a < b. \end{aligned} \tag{6}$$

Here K and E are elliptic integrals (for a sphere we have $H_m = \pi j_c a/4$), and for a cylinder (pellet) with radius a and thickness $2b$ we have

$$H_m^{\text{cyl}} = j_c b \ln[a/b + (1 + a^2/b^2)^{1/2}]. \tag{7}$$

If $a \gg b$, then $H_m^{\text{cyl}} \cong H_m^{\text{ell}} + j_c b(1 - \ln 2)$.

We note that if in Eqs. (3) and (4) the origin r_0 is treated as an independent variable, then the relations (4) become corollaries of (3). For a small displacement of r_0 by dz along the Z axis, the equation of the surface changes to $dR_0(u) = [(1 - u^2)\partial \ln R_0(u)/\partial u - u]\, dz$, and $R(u,H_e)$ also changes. It can be shown that relation (4) follows directly from the invariance of relation (3) under such transformations.

The infinitesimal form of relations (3) and (4) is useful for analyzing the evolution of the screened region:

$$\int D(u)\sqrt{1-u^2}\,du = 2, \tag{8}$$

$$\int D(u)\sqrt{1-u^2}\,P_n^{(1,1)}(u) R^{-n}(u,H_e)\,du = 0 \quad (n > 0), \tag{9}$$

where the rate of penetration of the magnetic flux into the sample

$$D(u) \equiv -j_c \partial R(u,H_e)/\partial H_e$$

is a function of the external field and a functional of $R(u,H_e)$. This function is linked with the Meissner current density $\mathbf{j}_{\text{meiss}}$ in a solid with the same shape as the screened region. Indeed, when the external field increases by a small amount dH_e, the current density can change only inside this region, since elsewhere in the sample it has the critical value. This change is such that the total induction here remains zero everywhere except in a thin layer near $r = R(u,H_e)$. Therefore, the current density changes by $d\mathbf{j} = \mathbf{j}_{\text{meiss}} dH_e$, where $\mathbf{j}_{\text{meiss}}$ is the Meissner current induced by a unit field. We note that, for example, the exact solution obtained in Ref. 2 for an infinitely thin film satisfies the relation $d\mathbf{j}/dH_e = \mathbf{j}_{\text{meiss}}$.

In the three-dimensional case, just as in the case of planar geometry,[2] the current $\mathbf{j}$ is thus a linear combination of Meissner currents. Now, however, each Meissner current is concentrated on the surface of the screened region and is expressed as $j_{\text{meiss}} = i(u)\delta(r - R(u,H_e))/\cos\alpha$. Here $i(u)$ is the corresponding surface current, induced by a unit external field, and α is the angle between the radius vector of a given point on the surface and the normal to the surface. It is obvious that the displacement of the surface, which is caused by an increase of H_e, into the sample along the direction of the normal is equal to $i(u)dH_e/j_c$ and the displacement along the radius vector is $1/\cos\alpha$ times greater. Expressing $\cos\alpha$ in terms of the equation of the surface, we therefore have

$$D(u) = i(u)[1 + (1 - u^2)(\partial \ln R(u)/\partial u)^2]^{1/2}. \tag{10}$$

The argument H_e is dropped to simplify the formulas.

We confine the analysis below to the case of reflection symmetry and we place the origin of coordinates at the center of the sample. If the sample is a sphere, then at the initial stage of the penetration of the field, i.e., for $R(u) \cong R_0(u) =$ const, as follows from Eq. (5), $D(u) = 3/2\sqrt{1-u^2} \sim \sin\vartheta$. Hence it is easy to see that the screened region immediately starts to taper at the poles and becomes spindle-shaped. It is obvious that such a formation of cones generally occurs at the poles. Because of this circumstance, as the field increases further, the tangential component of the induction at the poles becomes different from zero. This in turn results in a nonzero value of the surface Meissner current $i(u)$ at the poles ($u = \pm 1$) and, as one can see from Eq. (10), $D(\pm 1)$ also vanishes, i.e., it results in a gradual compression of the screened region along the Z axis.

3. Consequently, at least three parameters, which would determine two characteristic dimensions and the cone angles of the cones at the poles, are required for numerical modeling of the evolution of the critical state using an analytic approximation of the surface $R(u)$. The dependence of these parameters on the external field can be found from the three equations (3) and (4) with $n=2$ and $n=4$. It is interesting to compare the computational results obtained for a strongly oblate body with finite and variable thickness to the analytical expression obtained for an infinitely thin, circular disk.[2] For an ellipsoidal body it is natural to approximate the surface $r = R(u)$ in Cartesian coordinates by a curve obtained by a conical section. The corresponding expression in polar coordinates has the form $R(u) = b/[p(1-u^2)^{1/2} + (u^2 + s^2(1-u^2))^{1/2}]$, where b is the half-thickness of the screened region on the Z axis, the radius of the region is $a = b/(p+s)$, and the parameter $p \geqslant 0$ describes the tapering at the poles. Let $p=0$ and $a \gg b$ for $H_e = 0$. As expected, the decrease of the radius with increasing H_e is determined by the characteristic field $H_c = j_c b$. For $H_e > H_c$ and $H_m - H_e > H_c$, however, the numerically determined function $a(H_e)$ was found to be close to $a(H_e) = a(0)\exp(-H_e/H_c)$, while for a circular film $a(H_e) = a(0)/\cosh(H_e/H_c)$ with the same value of H_c; i.e., the radius is approximately two times longer. We note that the volume analog of a film is a very thin pellet (since in the pellet the integral of the critical current density over the thickness does not depend on the position in the horizontal plane). Therefore, a nonuniformity of the thickness results in a relatively small effective displacement of the external field by an amount of the order of $H_c = j_c b$ [i.e., as one can see from Eqs. (6) and (7), of the order of the difference between H_m^{cyl} and H_m^{ell}], although it markedly changes the radius of the screened region in the same field (by several tens of percent).

4. The results presented above refer to a monotonic increase of the external field starting from zero. This process is irreversible, since a subsequent decrease of the field leads not to retrograde motion of the screened region but rather to the appearance of a layer at the boundary of the sample and expansion of this layer with an inverted critical current (this corresponds to the vortices leaving the sample and nucleation of vortices of opposite polarity[1-4]). The geometry of the interface arising between the regions with oppositely directed currents can in principle be determined from the relations which extend the relations (1)–(4) in an obvious manner and the current configuration as a whole is constructed completely analogously to the way this was done in Refs. 2–4 for a film. However, approximate methods cannot be avoided. The analysis presented above could be helpful for implementing such methods.

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