# Fourth-order correlation function of a randomly advected passive scalar

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Advection of a passive scalar $\theta$ in $d=2$ by a large-scale velocity field rapidly changing in time is considered. The Gaussian feature of the passive scalar statistics in the convective interval was discovered in Ref. 1. Here we examine the deviations from the Gaussian behavior: The simultaneous fourth-order correlation function of $\theta$ is obtained analytically. Explicit expressions for fourth-order objects, like $\langle(\theta_1-\theta_2)^4\rangle$, are derived. © *1995 American Institute of Physics.*

Advection of a passive scalar $\theta$ by an incompressible turbulent flow is one of the classical problems in the theory of turbulence. This problem is related to statistics of the temperature or impurities in the flow. The dynamics of the passive scalar is governed by

$$(\partial_t+u_\alpha\nabla_\alpha-\kappa\Delta)\theta=\phi, \tag{1}$$

where the velocity $\mathbf{u}$ and the pumping $\phi$ are random functions of $t$ and $\mathbf{r}$, and $\kappa$ is the diffusion coefficient. The correlation functions of $\theta$ should be treated as averages over the statistics of $\phi$ and $\mathbf{u}$. Batchelor[2] was first to consider the problem of a long-range velocity field. He found the pair correlator of the passive scalar in the case where the velocity field is very slow. Kraichnan[3] considered the pair correlator in the opposite case of a velocity field which changes in time very rapidly. A theory for any finite correlation time of the velocity field was proposed in Ref. 1. It was proved there that, regardless of the statistics of the velocity field, the statistics of the passive scalar in the convective interval approaches Gaussian behavior as the Peclet number is increased (the ratio of the pumping scale to the diffusion scale).

In the present letter our goal is to find explicitly the fourth-order correlation function of the passive scalar in the Kraichnan regime. We assume that the source $\phi$ is $\delta$-correlated in time and spatially correlated on the scale $L$:

$$\langle\phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle=\delta(t_1-t_2)\chi_2(r_{12}), \tag{2}$$

  

where $\chi_2(r)$ tends to zero at $r \gg L$. We choose a simple, smooth form of the pumping

$$\chi_2(r) = P_2 L^2/(L^2 + r^2), \tag{3}$$

where $P_2$ is the production rate of $\theta^2$. A variation of the shape of $\xi_2(r)$ at $r > L$ will keep all the values of interest intact at $r \ll L$. The statistics of a velocity which is $\delta$-correlated in time is completely defined by the pair correlation function which for the dimension $d=2$ is

$$\langle u_\alpha(t_1, \mathbf{r}_1) u_\beta(t_2, \mathbf{r}_2) \rangle = \delta(t_1 - t_2)[D L_u^2 \delta_{\alpha\beta} - \mathscr{K}_{\alpha\beta}(\mathbf{r}_{12})], \tag{4}$$

$$\mathscr{K}_{\alpha\beta}(r) = D(3\delta_{\alpha\beta} r^2/2 - r_\alpha r_\beta), \tag{5}$$

where the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ is taken into account. Here $D$ is the characteristic strain which describes the strength of the velocity field, and $L_u$ is the velocity correlation length (the size of the largest vortex) which is assumed to be the largest scale in the problem. Expression (5) is correct at $r \ll L_u$.

Simultaneous correlation functions of $\theta$ satisfy separate linear equations.[4–6] The equation for the pair correlation function $f(|\mathbf{r}_1 - \mathbf{r}_2|) = \langle \theta_1 \theta_2 \rangle$ can be solved explicitly for arbitrary $r$. For the separations $r \gg r_d$, where $r_d = 2\sqrt{\kappa/D}$ is the so-called diffusive length, we obtain

$$f(r) = \frac{P_2}{2D}\{\ln(1 + L^2/r^2) + (L^2/r^2)\ln(1 + r^2/L^2)\}. \tag{6}$$

The equation for the fourth-order simultaneous correlation function of the passive scalar $\mathscr{F} = \langle \theta_1 \theta_2 \theta_3 \theta_4 \rangle$ is

$$-\hat{\mathscr{L}}\mathscr{F}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \Phi(r_{12}, r_{34}) + \Phi(r_{13}, r_{24}) + \Phi(r_{14}, r_{23}), \tag{7}$$

$$\hat{\mathscr{L}} = \sum_{i>j} \mathscr{K}_{\alpha\beta}(r_{ij}) \nabla_{i\alpha} \nabla_{j\beta} + \kappa \sum_{i=1}^{4} \Delta_i, \tag{8}$$

$$\Phi(r_+, r_-) = f(r_+)\chi_2(r_-) + f(r_-)\chi_2(r_+), \tag{9}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. Equation (7) can be generalized for an arbitrary multiscale velocity field and for an arbitrary dimension[5–7] $d$. Although at any $d$ the right side of (7) decomposes into three parts, each depending only on two separations, only in the case of the large-scale velocity field the operator $\hat{\mathscr{L}}$ supports the decomposed form: Acting on an arbitrary function of the two vectors, it produces a function of the two vectors. The corollary gives us a possibility to establish the following general form of the solution:

$$\mathscr{F}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = F(\mathbf{r}_{12}, \mathbf{r}_{34}) + F(\mathbf{r}_{13}, \mathbf{r}_{24}) + F(\mathbf{r}_{14}, \mathbf{r}_{23}). \tag{10}$$

Since we are looking for a solution in the restricted region $r \ll L_u$, we could add to (10) a zero mode of the operator $\hat{\mathscr{L}}$. However, the Eq. (7) stems from the dynamical approach (see Ref. 7 for details) which has no room for a solution other than the decomposed one. The function $F(\mathbf{r}_+, \mathbf{r}_-)$ from (10) satisfies the equation

$$-[\hat{\mathscr{L}}' + \hat{\mathscr{L}}_d] F(\mathbf{r}_+, \mathbf{r}_-) = \Phi(r_+, r_-), \tag{11}$$

$$\hat{\mathscr{L}}' = 2D \sin^2 \vartheta [\partial_y^2 + (\partial_\vartheta + \cot \vartheta \partial_\xi)^2], \quad \hat{\mathscr{L}}_d = 2\kappa(\Delta_+ + \Delta_-). \tag{12}$$

In (12) we used the variables $\xi=\ln[L^2/(r_+r_-)]$, $y=\ln[r_-/r_+]$, and $\vartheta=\arccos[\mathbf{r}_+\mathbf{r}_-)/(r_+r_-)]$. The physical boundary conditions imposed on $F$ are $F(\mathbf{r}_+,\mathbf{r}_-)\to 0$ at $r_+$ or $r_-\to\infty$.

As was shown in Ref. 1, the correlators of $\theta$ do not sense a diffusion if $r_\pm \gg r_d$. Thus, at $r\gg r_d$ we can omit the diffusive term in (11) which together with the independence of $\Phi(r_+,r_-)$ on $\vartheta$, allows us to write a solution of (11) as the sum $F=F_++F_-$, where $F_\pm$ satisfy the equations

$$-2D\sin^2\vartheta[\partial_\vartheta^2+\partial_y^2]F_\pm=f(r_\pm)\chi_2(r_\mp). \tag{13}$$

Here $r_\pm=\sqrt{S/|\sin\vartheta|}\exp(\mp y/2)$ and $S$, $\vartheta$, and $y$ should be treated as independent parameters. We conclude that by construction we have $F_+(S,\vartheta,y)=F_-(S,\vartheta,-y)$. Taking into account the symmetry properties of (13), we find that $F_\pm$ is invariant under $\vartheta\to-\vartheta$ and under $\vartheta\to\pi-\vartheta$. To solve Eq. (13), we can use the resolvent $\mathscr{R}$ of the Laplacian $\partial_\vartheta^2+\partial_y^2$ in (13):

$$-[\partial_\vartheta^2+\partial_y^2]\mathscr{R}(\vartheta,\vartheta',y-y')=\delta(\vartheta-\vartheta')\delta(y-y'). \tag{14}$$

We should impose the zero boundary conditions on the resolvent at the boundaries of the strip: $0<\vartheta<\pi$ and $-\infty<y<+\infty$, since in accordance with the definition $r_\pm\to\infty$ if $\vartheta\to 0$, $\pi$ or $y\to\pm\infty$. The resolvent can be written explicitly:

$$\mathscr{R}=\frac{1}{4\pi}\ln\left[\frac{\sinh^2(y/2-y'/2)+\sin^2(\vartheta/2+\vartheta'/2)}{\sinh^2(y/2-y'/2)+\sin^2(\vartheta/2-\vartheta'/2)}\right]. \tag{15}$$

Convolution of the right side of (13) with the resolvent gives

$$F_+=\frac{L^2P_2}{4Dr_-^2\sin^2\vartheta}\int_0^\infty dwf(r_+w)\tau^{-1}\ln\left[\frac{(w_++\tau)^2+w^2\cot^2\vartheta}{(w_-+\tau)^2+w^2\cot^2\vartheta}\right], \tag{16}$$

where $w_\pm=w\pm w^{-1}$, and $\tau=\sqrt{L_2r_-^{-2}\sin^{-2}\vartheta+w^{-2}}$. Note that this formula leads to the weak angular singularity of $F_+$ at $\vartheta=0$:

$$\partial_\vartheta F_+|_{\vartheta=0}=\pi P_2^2/D^2(e^{2y}\ln(1+e^{-y})-e^y)\neq 0. \tag{17}$$

Below we will show that the diffusion smoothes the singularity at the smallest angles. In general, (16) substituted into $F=F_++F+(y\to-y)$ and then into (10) closes the problem of finding the fourth-order correlation function $\mathscr{F}$ of the passive scalar for all separations larger $r_d$.

For $L\gg r_\pm$ the integration in Eq. (16) can be performed. We then have

$$F=\frac{P_2^2}{D^2}\{\ln^2[L/r_+]/2+\ln^2[L/r_-]/2+\ln[L/r_+]+\ln[L/r_-]+\pi^2/12\}+\mathscr{Y}(z)$$
$$+\mathscr{Y}(\bar{z})+\mathscr{O}(r_\pm/L), \tag{18}$$

$$\mathscr{Y}(z)=\frac{P_2^2}{2D^2}\{\pi\,\mathrm{Im}[e^{-iz}]-\mathrm{Re}[\Omega(-ie^{iz})]-\mathrm{Re}[e^{-2iz}\Omega(ie^{-iz})]\},$$

$$\Omega(u)=\frac{\pi^2}{12}+\ln^2u+\mathrm{Li}_2(1-iu)+\mathrm{Li}_2(1+iu), \tag{19}$$

where $z=\vartheta+iy$, and $\bar{z}=\vartheta-iy$, and Li in (19) is the integral logarithm. Let us emphasize that the second part of (18), $\propto[\mathcal{Y}(z)+\mathcal{Y}(\bar{z})]$, turns out to be a zero mode of the Laplacian, since it is the real part of an analytic function of $z$. The general result (18) shows a remarkable coincidence of $F$, within a logarithmic accuracy, with its collinear limit $F(\vartheta=0)$. It follows from the dynamic approach described in Ref. 1. It was demonstrated there that $F$ is an average over velocity's statistics of the product of times required for the vectors $\mathbf{r}_+$ and $\mathbf{r}_-$, respectively, to grow to $L$ if their edge points move along the Lagrangian trajectories. An evolution of the vectors which are originally deep inside the convective interval $r_\pm \ll L$, can be divided into two different stages: The first stage is a quick collinearization of the geometry and second one is a long collinear stretching which gives the major contribution to the correlator. For $L\gg r_+ \gg r_-$, (18) gives

$$[F-f(r_+)f(r_-)]\approx\frac{P_2^2}{D^2}\{\ln[L/r_+]+1+\vartheta^2-\pi|\vartheta|/2\}. \tag{20}$$

On the left side of this equation we subtracted from $F$ the major Gaussian part.

Let us now return to $F_+$ to examine its behavior at small angles. The angular singularity (17) is formed by the $z$-dependent part of (16) which is a zero mode of $\hat{\mathcal{L}}'$. At small angles it should be replaced by a zero mode $\mathcal{Z}$ of $\hat{\mathcal{L}}+\hat{\mathcal{L}}_d$, where the diffusive operator $\hat{\mathcal{L}}_d$ can be rewritten in terms of the set of variables $(\vartheta,y,\xi)$ as follows:

$$\hat{\mathcal{L}}_d=2\kappa\frac{e^{\xi}}{L^2}\cosh[y]\{\partial_\xi^2+\partial_y^2+\partial_\vartheta^2+4\ \tanh[y]\partial_\xi\partial_y\}. \tag{21}$$

The angular singularity is formed by the $z$-dependent part of (16) which an be treated as a zero mode of $\hat{\mathcal{L}}'$ by analogy with (18). At small angles it should be replaced by a zero mode $\mathcal{Z}$ of the sum $\hat{\mathcal{L}}'+\hat{\mathcal{L}}_d$. We are looking for a zero mode $\mathcal{Z}$ which is proportional to $|\vartheta|$ at values of the angle, where it is possible to ignore the effect of diffusion at all, and is regular in the limit $\vartheta\to 0$. For the smallest value of $\vartheta$, $\partial_\vartheta\mathcal{Z}$ is much larger than $\partial_y\mathcal{Z}$ and $\partial_\xi\mathcal{Z}$. Thus, $\mathcal{Z}$ should satisfy

$$\{(\vartheta^2\partial_\vartheta^2+2\vartheta\partial_\vartheta\partial_\xi-\partial_\xi+\partial_\xi^2)+Ae^{\xi}\partial_\vartheta^2\}\mathcal{Z}=0, \tag{22}$$

where $A=\kappa\cosh(y)/(16DL^2)$ can be considered as an arbitrary parameter. Taking into account (17), we find the following solution of (22) which has the desirable behavior:

$$\mathcal{Z}\approx\pi\frac{P_2^2}{D^2}[e^{2y}\ \ln(1+e^{-y})-e^{y}]\sqrt{\vartheta^2+\kappa\ \cosh\ ye^{\xi}/(DL^2)}. \tag{23}$$

The characteristic angle $\vartheta_0=\sqrt{\kappa D^{-1}L^{-2}\ \cosh[y]}\exp(\xi/2)$ turns out to be small at $r_\pm\gg r_d$. Using the explicit form (23), we conclude that for $\vartheta\sim\vartheta_0$ we have $\partial_\vartheta\mathcal{Z}\sim\vartheta_0^{-1}\mathcal{Z}$, $\partial_y\mathcal{Z}\sim\mathcal{Z}$, and $\partial_\xi\mathcal{Z}\sim\mathcal{Z}$. These estimates justify the calculations we have given above. In summary, for $r_\pm\gg r_d$ the diffusion is relevant only at small angles $\vartheta<\vartheta_0$, where it influences the angular derivatives of $F$, but it gives a negligible correction to expression (16).

As long as we consider a correlation function at sufficiently small distances, say $F$ at $r_-<r_d$, the diffusion must be taken into account. The major value of the function can be found directly from a suitable expression in the convective interval by inserting there, in

terms formally divergent at $r \to 0$, $r_d$ instead of the smallest distances (see Appendix A of Ref. 1 for the proof). Thus, using (18), we find with the logarithmic accuracy the following fourth-order objects:

for $L \gg r_{13}, r_{14}, r_{23}, r_{24} \gg r_{12} > r_{34} \gg r_d$, $\quad r_{12} \| r_{34}$,

$$\langle(\theta_1-\theta_2)^4\rangle \approx 12\, \frac{P_2^2}{D^2}(\ln^2[r_{12}/r_d]+\ln[r_{12}/r_d]), \tag{24}$$

$$\langle(\theta_1-\theta_2)^2\theta_3^2\rangle-\langle(\theta_1-\theta_2)^2\rangle\langle\theta_3^2\rangle \approx 2\, \frac{P_2^2}{D^2} \ln[r_{12}/r_d], \tag{25}$$

$$\langle(\theta_1-\theta_2)^2(\theta_3-\theta_4)^2\rangle-\langle(\theta_1-\theta_2)^2\rangle\langle(\theta_3-\theta_4)^2\rangle \approx 4\, \frac{P_2^2}{D^2} \ln[r_{34}/r_d]. \tag{26}$$

To switch from correlators (25) and (26) to respective correlators of the dissipative field $\epsilon=\kappa(\nabla\theta)^2$, we should differentiate them over $r_{12}, r_{34}$ and replace the separations by $r_d$. We obtain the estimates

for $L \gg r_{13} \gg r_d$ $\quad \langle\langle\epsilon_1\theta_3^2\rangle\rangle=\kappa\langle\langle(\nabla\theta_1)^2\theta_3^2\rangle\rangle \sim P_2^2/D, \quad (27)$

$$\langle\langle\epsilon_1\epsilon_3\rangle\rangle=\kappa^2\langle\langle(\nabla\theta_1)^2(\nabla\theta_3)^2\rangle\rangle \sim P_2^2. \tag{28}$$

The unknown multipliers behind the parametric dependences in Eqs. (27) and (28) are on the order of unity. We postpone an explicit calculation of the multipliers, which requires a direct account of diffusion. Equations (27) and (28) show (in accordance with Ref. 7) the zero dimensionality for the passive scalar $\theta \sim r^0$ and for the dissipation field $\epsilon \sim r^0$.

Note that in the region $L_u \gg r_{ij}, r_{kl} \gg L, r_{ik}, r_{il}, r_{jk}, r_{jl}$ the correlator $\mathcal{F}$ which was found does not decay. The decay occurs only for the largest scales $r_{12}, r_{34} \gg L_u$, which are outside the scope of the present study.

The proposed solution for the fourth-order correlation function of $\theta$ could serve as a starting point for studying passive scalar correlations in the general case of a multiscale (although short-correlated in time) velocity field. This problem was introduced by Kraichnan.[8] A closure[5,9] which was applied recently to the model yields an anomalous scaling, particularly for $\langle(\theta_1-\theta_2)^4\rangle$ in any space dimensions, while the consideration given in Ref. 7 shows the normal scaling for the object at $d \geq 3$. Future attempts of a perturbative study based on the results of the present letter could solve the collision at $d=2$. On the other hand, inclusion of the present results into the scheme proposed in Refs. 10 and 11 could help better understand the problem of the direct cascade in two-dimensional turbulence.

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