Three nondissipative forces on a moving vortex line in superfluids and superconductors

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In the hydrodynamic limit the nondissipative force F_{nd} that acts on a vortex in a superconductor or a Fermi superfluid, contains three different contributions of topological origin. These are (i) the Magnus force, (ii) the Iordanskii force, whose origin is analogous to the Aharonov–Bohm effect for spinning cosmic strings, and (iii) the force resulting from the spectral flow of fermion zero modes in the vortex core, which leads to the production of quasiparticle momentum when the vortex moves with respect to the normal component. The latter force leads to an anomaly in the thermodynamics of the moving vortices; the possible relationship of this anomaly to the Unruh effect is discussed. © *1995 American Institute of Physics.*

1. General expression for the nondissipative forces

Recent developments in high-T_c superconductivity and in the physics of superfluid ^{3}He have renewed interest in the dynamics of vortices in these pair-correlated Fermi systems.[1–5] The theoretical investigation of the vortex dynamics was started 3 decades ago (see the classic papers in Refs. 6–9). In the linear regime this dynamics was finally calculated by Kopnin and coauthors.[1,10,11] However the correct interpretation of the nondissipative forces acting on a moving vortex line was missing.

Here we consider the hydrodynamic limit $\omega_0\tau \ll 1$, where ω_0 is the distance between the levels of the fermions localized within the vortex core[12] and τ the lifetime of the fermions on these levels. In this case the dissipative force can be neglected and we also neglect the pinning forces. Our statement is that in this limit the nondissipative force $\mathbf{F}_{nd}$ that acts on a vortex contains three different contributions. $\mathbf{F}_{nd}$ is expressed in terms of 3 different combinations of the relevant velocities: the velocity $\mathbf{v}_L$ of the vortex line, the velocity $\mathbf{v}_s$ of the superfluid vacuum (or superfluid velocity), and the velocity $\mathbf{v}_n$ of the normal component of the liquid (the velocity of the heat bath):

$$\mathbf{F}_{nd}=\mathbf{F}_{\rm Magnus}+\mathbf{F}_{\rm Iordanskii}+\mathbf{F}_{\rm spectral\ flow}, \quad \mathbf{F}_{\rm Magnus}=\vec{\kappa}\times\rho(\mathbf{v}_L-\mathbf{v}_s),$$
$$\mathbf{F}_{\rm Iordanskii}=\vec{\kappa}\times\breve{\rho}_n(T)(\mathbf{v}_s-\mathbf{v}_n), \quad \mathbf{F}_{\rm spectral\ flow}=\vec{\kappa}\times C_0(\mathbf{v}_n-\mathbf{v}_L). \tag{1}$$

Each of the three forces is of topological origin.

(i) According to the Landau picture of a superfluid its motion consists of the motion of the superfluid vacuum (with total mass density ρ and superfluid velocity $\mathbf{v}_s$) and the dynamics of the elementary excitations. The Magnus force in Eq. (1) acts on a vortex if it moves with respect to the superfluid vacuum. Here κ is the circulation vector: for Fermi (Bose) superfluids $\kappa=\pi N\hbar/m$ ($\kappa=2\pi N\hbar/m$), where N is the vortex winding number and m is the bare mass of the fermion (boson). The Magnus force comes from the flux of linear momentum from the vortex to infinity. The topological origin of this force was discussed in Refs. 3, 4.

(ii) The Iordanskii force[9,11] results from elementary excitations outside the vortex core: the vortex line produces for them an Aharonov–Bohm potential. This force can be obtained as a sum of the forces acting on the individual particles according to the equation $\partial_t \mathbf{p}=(\vec{\nabla}\times\mathbf{v}_s)\times\mathbf{p}$, where $\mathbf{p}$ is the quasiparticle momentum and the vorticity $\vec{\nabla}\times\mathbf{v}_s=\vec{\kappa}\delta_2(\mathbf{r})$ is concentrated in a thin tube (vortex core). The Iordanskii force is thus

$$-\sum_{\mathbf{p}} \partial_t \mathbf{p}=-\vec{\kappa}\times\int d^3p/(4\pi^3) f\,[E_{\mathbf{p}}+\mathbf{p}\cdot(\mathbf{v}_s-\mathbf{v}_n)]=\vec{\kappa}\times\check{\rho}_n(T)(\mathbf{v}_n-\mathbf{v}_s).$$

Here f is a Fermi or Bose function, depending on the type of elementary excitations, and is Doppler shifted due to the counterflow $\mathbf{v}_n-\mathbf{v}_s$; $\tilde{\rho}_n(T)$ is the density of the normal component, which can be an anisotropic tensor. The Iordanskii force is the only nondissipative force in Eq. (1) that depends on temperature T. In Section 2 we discuss the analogy of the Iordanskii effect with the Aharonov–Bohm effect for spinning cosmic strings.[13]

(iii) The third term exists only in fermionic systems. It results from momentum exchange between the fermions localized in the vortex core and the fermions in the heat bath. It is described by the spectral flow of the fermion zero modes within the vortex core and is related to the Callan–Harvey mechanism of anomaly cancellation applied to the fermion zero modes on vortices.[2] In Ref. 2 the parameter C_0 was obtained in the zero-temperature limit, where for a spherical Fermi surface it is expressed in terms of the Fermi momentum p_F: $C_0=mp_F^3/3\pi^2$. It was found recently that the spectral flow is unaffected by T, since the temperature does not change the topology of the spectrum of fermion zero modes, and thus the parameter C_0 equals its zero-temperature value.[14] This finding allowed us to construct a general expression for the nondissipative force $\mathbf{F}_{nd}$ in Eq. (1) in the limit $\omega_0\tau\ll 1$, in which the spectral flow of the momentum is not suppressed.

In Section 3 we consider the anomaly in the thermodynamics of the mixed state (state with vortices) due to the spectral flow contribution to the nondissipative force, which can lead to an effective temperature of the moving vortex and to an analog of the Hawking–Unruh radiation[15,16] (Section 4).

If one adds the dissipative force to Eq. (1) one obtains the conventional expression for the balance of forces acting on the vortex

$$\rho_s(\mathbf{v}_s-\mathbf{v}_L)\times\vec{\kappa}+D\kappa(\mathbf{v}_n-\mathbf{v}_L)-D'(\mathbf{v}_n-\mathbf{v}_L)\times\vec{\kappa}=0. \tag{2}$$

Here $\rho_s=\rho-\rho_n$; the parameters D and D' were calculated by Kopnin; the factor D in the dissipative force is small in the limit $\omega_0\tau\ll 1$.[1] From a comparison with Eq. (1) it follows that the parameter $D'=C_0-\rho_n$. This is slightly different from Kopnin's result $D'=\rho_s$, which follows if one neglects the difference between the anomaly parameter C_0 and the mass density ρ. In conventional superconductors and in superfluid ^{3}He the difference between C_0 and ρ is very small, of order $(\Delta/E_F)^2$, where Δ is the gap in the quasiparticle spectrum and E_F is the Fermi energy. However, one can imagine systems where this difference is sizable.

2. Aharonov–Bohm effect and an analog of the spinning string

To clarify the analogy between the Iordanskii force and the Aharonov–Bohm effect, let us consider the simplest cases of phonons propagating in the velocity field of a quantized vortex in the Bose superfluid ^{4}He and fermions propagating in the velocity field of a quantized vortex in the Fermi superfluid ^{3}He-A. According to Unruh[17] the dynamics of the phonons in the presence of the velocity field is the same as the dynamics of photons in the gravity field. For the velocity field of a quantized vortex the phonons obey the equation of motion of a scalar wave in the metric

$$ds^2=(c^2-v^2(r))\left(dt+\frac{\kappa}{2\pi(c^2-v^2(r))}\,d\phi\right)^2-dr^2-dz^2-\frac{c^2}{c^2-v^2(r)}\,r^2d\phi^2. \quad (3)$$

Here c is the speed of sound, $\mathbf{v}=\hat{\phi}\kappa/2\pi r$ is the superfluid velocity around the vortex, and we use a cylindrical system of coordinates with its z axis along the vortex line. The same metric takes place for the gapless Bogoliubov fermions propagating in the field of the axisymmetric phase vortex in the A phase of superfluid ^{3}He (see Eq. (4.5) in Ref. 18). In this case the "velocity of light" is anisotropic and its transverse component is $c=\Delta/p_F$, where Δ is the amplitude of the gap.

Far from the vortex, where $v(r)$ is small and can be neglected, this metric corresponds to that of the so called rotating cosmic string. A spinning cosmic string (see the latest references[19,20]) is such a string which has rotational angular momentum. The metric in Eq, (3) corresponds to a string with angular momentum $J=\kappa/8\pi G$ per unit length and with zero mass.

An effect peculiar to the spinning string is the gravitational Aharonov–Bohm topological effect.[13] Though the metric outside the string is flat, there is a time difference for particles propagating around the spinning string in opposite directions. For a vortex (at large distances from the core) this time delay approaches

$$2\tau=2\kappa/c^2. \quad (4)$$

This asymmetry between particles moving on different sides of a vortex is the ultimate cause of the Iordanskii force acting on a vortex in the presence of a net momentum of the quasiparticles: $2\int d^3p/(2\pi)^3\mathbf{p}f(\mathbf{p})$.

3. Anomaly in the low-temperature hydrodynamics

The parameter C_0 in Eq. (1) results in an anomaly in the hydrodynamics of a rotating liquid, which becomes pronounced at low temperature, $T\to 0$ (i.e., $T\ll T_c$). Actually we consider an intermediate asymptote at which the system is still in the hydrodynamic regime and the condition $\omega_0\tau\ll 1$ is satisfied. This intermediate limit of $T\ll T_c$ can be attained, for example, in ^{3}He-A,[1] where ω_0 is extremely small for continuous vortices, and also in superconductors for which the lifetime τ is governed by impurities.

Let us consider a state in which the normal and superfluid components rotate with different angular velocities:

$$\mathbf{v}_s=\vec{\Omega}_s\times\mathbf{r},\quad \mathbf{v}_n=\vec{\Omega}_n\times\mathbf{r}. \tag{5}$$

For the superfluid component the uniform (an average) rotation is modeled by a system of identical rectilinear vortices with density $n=(2/\kappa)\Omega_s$. These vortices rotate with the velocity $\mathbf{v}_L$ found from the force balance Equation (2). The equilibrium situation with $\Omega_n\neq\Omega_s$ is possible because we neglect dissipation ($D=0$), and so there is no friction between the normal and the superfluid rotations.

An immediate consequence of the nonzero parameter C_0 is that in addition to the radial pressure gradient

$$P(r)-P(0)=\frac{1}{2}\,\rho_s v_s^2(r)+\frac{1}{2}\,\rho_n v_n^2(r), \tag{6}$$

the hydrodynamic equations for rotating superfluids (see Rev. 21) imply that there is also a temperature gradient

$$-S\partial_r T=\partial_r\{v_s[\rho v_L-(\rho_n v_n+\rho_s v_s)]\}, \tag{7}$$

where S is the entropy density of the rotating liquid. This is the so-called thermorotation effect,[21] and it takes place when the vortex velocity deviates from the center-of-mass velocity $(\rho_n v_n+\rho_s v_s)/\rho$ (see also the discussion in 22, where a variant of this effect was observed in rotating ^{3}He-B and was used for an experimental estimation of the parameters D and D'). As follows from Eqs. (2) and (7), in the absence of D the thermorotation effect comes solely from the spectral flow

$$-S\partial_r T=C_0\partial_r[v_s(v_L-v_n)]=2C_0\Omega_s(v_L-v_n))=\hbar n\,\frac{p_F^3}{3\pi}\,(v_L-v_n). \tag{8}$$

The right-hand side of Eq. (8) is temperature independent and thus is valid even at the $T\to 0$ limit. This could mean that even in this limit the moving (rotating) vortices have a nonzero entropy and a nonzero intrinsic temperature. The effect is governed by the spectral flow of fermions from the vortex into the bulk liquid. Thus T and S are related to the radiation of the fermions from the vortex, if the vortex moves relative to the heat bath, i.e., if $v_L\neq v_n$. This is reminiscent of the Unruh effect in quantum field theory:[16] an object moving with constant acceleration a radiates particles as a black body with temperature $T_{\rm Unruh}=\hbar a/2\pi c$. This results from an apparent change of the vacuum state, as measured by an accelerating particle detector. The main difference is that in superfluids the vacuum state can change even if the object moves with constant velocity: this leads to the radiation of particles if, say, a critical velocity is exceeded.

If one considers this analogy seriously, the moving vortex should have an effective temperature, which governs the radiation of the fermions by the moving vortex. This temperature corresponds to the average energy of the fermion zero mode in the vortex as measured in the frame of the heat bath:

$$T_{\rm eff}=\overline{|(\mathbf{v}_L-\mathbf{v}_n)\cdot\mathbf{p}|}=\frac{2}{\pi}\,|\mathbf{v}_L-\mathbf{v}_n|p_F\int_0^{\pi/2}d\phi\,\cos\phi=\frac{2}{\pi}\,|\mathbf{v}_L-\mathbf{v}_n|p_F. \tag{9}$$

4. Tunneling of particles and effective temperature

Let us consider the regime in which $\omega_0\tau$ is large, and the discrete nature of the generalized angular momentum Q which describes the fermions in the vortex core becomes important. In this case the spectral flow is suppressed and occurs only due to tunneling of the particles between discrete levels as a result of the vortex motion. The exponential suppression of the forces acting on a vortex was recently discussed in Ref. 5, which corresponds to the effective temperature in Eq. (9) but without the factor $2/\pi$. We show now that in fact a factor of $2/\pi$ was overlooked in Ref. 5 and that the tunneling rate between the neighboring levels is governed by $\exp -(\omega_0/T_{\text{eff}})$.

The Hamiltonian which describes the problem at low T is related only to the low-energy anomalous branch of spectrum, i.e., to the fermion zero mode:

$$\mathcal{H}=\mathbf{Q}\omega_0+\omega_0 t(\mathbf{v}_L\times\mathbf{p})\cdot\hat{z}. \tag{10}$$

Here $\mathbf{Q}$ is the generalized angular momentum operator for rotation about the vortex axis z[12] and $\mathbf{p}$ is the linear momentum operator for the particle. For simplicity we consider the 2-dimensional case, i.e., when ω_0 does not depend on p_z. The second term comes from the change of the angular momentum if it is observed in the heat-bath frame, $\mathbf{Q}\to\mathbf{Q}+\mathbf{r}(t)\times\mathbf{p}$, and $\mathbf{r}(t)=(\mathbf{v}_L-\mathbf{v}_n)t$ (further, we choose $\mathbf{v}_n=0$). The operator $\mathbf{Q}$ and the transverse linear momentum operator $\mathbf{p}=\mathbf{p}_\perp$ do not commute:

$$[\mathbf{Q},\mathbf{p}]=i\hat{z}\times\mathbf{p}. \tag{11}$$

In terms of the matrix elements between the states with different Q the Hamiltonian is:

$$\begin{gathered}\mathcal{H}_{QQ'}=Q\delta_{QQ'}\omega_0+\omega_0 t(\hat{z}\times\mathbf{v}_L)\langle Q|\mathbf{p}|Q'\rangle,\\ \langle Q|\mathbf{p}|Q'\rangle=\frac{1}{2}p_F((\hat{y}+i\hat{x})\delta_{Q,Q'+1}+(\hat{y}-i\hat{x})\delta_{Q,Q'-1}),\end{gathered} \tag{12}$$

where we have used the condition $p^2=p_F^2$.

If the vortex velocity v_L is small compared to ω_0/p_F, the semiclassical approach becomes valid. In this case the level flow is determined by the exponentially small transition probability between two neighboring levels. Let us find this exponent. The Hamiltonian for the two-level system, Q and $Q+1$, is

$$\mathcal{H}=\left(Q+\frac{1}{2}\right)\omega_0+\frac{1}{2}\omega_0\begin{pmatrix}1 & v_L t p_F\\ v_L t p_F & -1\end{pmatrix}. \tag{13}$$

The square of the energy reckoned from the midpoint between these two states is

$$\left[E-\left(Q+\frac{1}{2}\right)\omega_0\right]^2=\frac{1}{4}\omega_0^2[1+(v_L t p_F^2)]. \tag{14}$$

The trajectory $t=i\tau$ which connects two states along the imaginary time axis gives the following transition probability between the states in the exponential approximation:

$$w\propto \exp-2\ \text{Im}\ S,\quad \text{Im}\ S=2\int_0^{\tau_0}d\tau\frac{1}{2}\omega_0\sqrt{1-\frac{\tau^2}{\tau_0^2}},\quad \tau_0=\frac{1}{v_L p_F}. \tag{15}$$

This gives

$$w \propto \exp - \frac{\pi}{2} \frac{\omega_0}{v_L p_F}. \tag{16}$$

Thus the tunneling rate is equivalent to a thermal distribution of quasiparticles on the levels of the anomalous branch of the spectrum, with the effective temperature given in Eq. (9).

5. Discussion

We found three topologically different contributions to the nondissipative force acting on a moving vortex. The third contribution, which comes from the spectral flow, is anomalous: it reflects the Callan–Harvey effect of anomaly cancellation and leads to a low-temperature anomaly in the vortex thermodynamics. This anomaly can be described by some effective temperature of the vortex cores in Eq. (9), which governs the radiation of fermions from the moving vortex. If Eq. (9) is taken seriously, then it follows from Eq. (8) that there is an entropy density of the system of vortices related to the spectral flow:

$$S(r) = -m_3 \frac{p_F^2}{3\pi} \Omega_s r = -\frac{1}{6} p_F^2 n r. \tag{17}$$

This corresponds to an entropy per vortex $\frac{1}{6} p_F^2 r L$, where L is the length of the vortex, or $\frac{1}{6} p_F^2 A$, where A is the area of the surface between the vortex line and the axis of rotation.

The effect is possibly somewhat similar to the temperature and entropy of a black hole. For a black hole[15] the entropy is related to the process of crossing the surface of the hole's horizon and equals the area A of the horizon multiplied by some fundamental constant (for the black hole this is the square of the Planck momentum): $S = (1/4) A p_{\mathrm{Planck}}^2$.[15] In our case the Planck momentum is replaced by the Fermi momentum, and also the factor 1/6 is different from the black hole value 1/4 which may reflect the difference in the cut-off geometry.

A possible interpretation of the area law for the vortex entropy is that it derives from the multivaluedness of the condensate phase around the vortex, which according to Eq. (3) produces an Aharonov–Bohm jump in the phase of the quasiparticle wave function on some surface bounded by the vortex loop.

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