Self-focusing instability of two-dimensional solitons and vortices

E. A. Kuznetsov
Landau Institute for Theoretical Physics, Russian Academy of Sciences, 117334 Moscow, Russia

J. Juul Rasmussen
Department of Optics and Fluid Dynamics, Risö National Laboratory, DK-4000, Roskilde, Denmark

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The instability of two-dimensional solitons and vortices is demonstrated in the framework of the three-dimensional nonlinear Schrödinger equation (NLSE). The instability can be regarded as the analog of the Kadomtsev–Petviashvili instability [B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970)] of one-dimensional acoustic solitons in media with positive dispersion. At large distances between vortices this instability transforms into the Crow instability [S.C. Crow, AIAA J. **8**, 2172 (1970)] of two vortex filaments with opposite circulations. © *1995 American Institute of Physics.*

1. The Kadomtsev–Petviashvili (KP) instability[1] of one-dimensional acoustic solitons was the first in a set of the analogous instabilities of solitons in hydrodynamics and nonlinear optics (see, for example, Refs. 2–4). The cause of this instability is quite general (see Ref. 5). An acoustic soliton in a medium with positive dispersion represents a propagating density well. The amplitude (the well depth) decreases with increasing velocity. Therefore, upon transverse modulation of the soliton the regions with smaller amplitude (shallow wells) will overtake those with higher amplitude (deep density wells). This results in an instability of the self-focusing type. In this Letter we demonstrate that the instability of two antiparallel vortex filaments predicted first by Crow[6] for ideal fluids and the KP instability of acoustic solitons can be regarded as two different limits of the same instability for the whole family of two-dimensional (2D) solitons and vortices described by the nonlinear Schrödinger equation (NLSE) with repulsion:

$$i\psi_t+\tfrac{1}{2}\nabla^2\psi+(1-|\psi|^2)\psi=0. \tag{1}$$

As is well known, this equation has at least two important applications. The first one relates to nonlinear optics, where Eq. (1) describes the propagation of electromagnetic waves in defocusing media when the refractive index has a negative nonlinear addition proportional to the light intensity ($\sim|\psi|^2$). In this case the nonlinear term amplifies the linear effects, diffraction and dispersion, by broadening the optical pulse in the transverse and longitudinal (relative to the propagation of the pulse) directions. Thus, meaningful nonlinear dynamics is only possible for pulses sufficiently long in time and wide in the transverse direction, when, for instance, dark solitons are observed. Therefore we will further assume that ψ tends to a constant value, say, to 1, as $|\mathbf{r}| \to \infty$. In such a formulation Eq. (1) is also used as a model for the description of the condensate motion in a weakly imperfect Bose gas, with ψ being the condensate wave function. For the Bose gas this equation was first derived by Gross[7] and Pitaevskiĭ[8] and it is therefore sometimes called the Gross–Pitaevskiĭ equation.

2. Depending on the spatial dimensions of the problem, the NLSE (1) gives rise to different nonlinear behaviors. As is well known, in the one-dimensional case the equation can be integrated by the inverse scattering transform[9]. One of the main results of this theory is a proof of the stability of one-dimensional solitons. In optics such objects are called grey solitons, and, accordingly, ones at rest are called dark solitons.

In two and three dimensions soliton solutions cannot be found explicitly in the entire range of parameters (except for some limited cases), but only numerically. Multidimensional solitons have been studied in detail in several papers, mainly in the context of the dynamics of the Bose condensate. Of particular note among these are a series of papers by Roberts and co-authors[10,11] and the paper of Iordanskii and Smirnov.[12]

The shape of the soliton solution in the form $\psi=\psi_0(x-vt,r_\perp)$ (here we are considering only axisymmetric solutions) is determined by integration of the equation

$$-iv\,\frac{\partial\psi_0}{\partial x}+\frac{1}{2}\,\nabla^2\psi_0+(1-|\psi_0|^2)\psi_0=0. \tag{2}$$

Here v is the velocity of the soliton, and $\psi\to 1$ for all directions as $r\to\infty$. It is easy to see that this solution (as well all other stationary ones) can be obtained from the following variational problem:

$$\delta(H-vP_x)=0, \tag{3}$$

where

$$H=\frac{1}{2}\int\,[|\nabla\psi|^2+(|\psi|^2-1)^2]d\mathbf{r}, \tag{4}$$

$$\mathbf{P}=\int\,n\mathbf{U}d\mathbf{r} \tag{5}$$

are the Hamiltonian and the momentum,[1)] respectively. Here we introduce the density fluctuation $n=N-1$ and the velocity $\mathbf{U}=\nabla\phi$, related to the wave function as $\psi=\sqrt{N}e^{i\phi}$.

Equation (3) says that the soliton solution represents a stationary point of the Hamiltonian for fixed momentum. The Lagrange multiplier v in (3) coincides with the soliton velocity in (2). Hence, in particular, it follows that on the soliton family the velocity v can be defined as

$$v=\frac{\partial\varepsilon}{\partial P}, \tag{6}$$

where ε is the soliton energy and $P=P_x$ is the x component of its momentum. The range of possible soliton velocities is determined by the form of the spectrum of small oscillations against the background of constant density, $N=1$; for Eq. (1) (the Bogolyubov spectrum), $\omega = k(1 + k^2/4)^{1/2}$. It has to lie in the interval between 0 and the minimum phase velocity $v_{\rm ph} = \omega/k$, coinciding with the sound velocity $C_s=1$. The soliton velocity cannot exceed the minimum phase velocity because then the Cherenkov-like radiation will become possible, and, as a result, such a localized structure cannot be stationary—it will lose its energy and finally disappear. Therefore, close to the threshold for Cherenkov radiation, but for $v<C_s$, the amplitude of the soliton will be small, and it will vanish for $v=C_s$. Near the threshold the nonlinearity, being weak on the soliton solution, is compensated by the (positive) dispersion, which also has to be weak for this reason. In this velocity region in the 2D case the soliton solutions are close to the 2D acoustic solitons of the KPI equation. The regular procedure of such a reduction from the NLSE (1) to the KP equation consists in the introduction of both slow time and slow coordinates, $t'=\varepsilon^3 t$, $x'=\varepsilon(x-C_s t)$, $y'=\varepsilon^2 y$, $z'=\varepsilon^2 z$, and the representation of n in the form of a series in powers of the small parameter ε (for stationary solitary waves $\varepsilon = \sqrt{1-v}$). The KP equation appears in third order ($\sim\varepsilon^3$),

$$\frac{\partial}{\partial x}\left(n_t-\frac{1}{8}n_{xxx}+\frac{3}{2}nn_x\right)=-\frac{1}{2}\nabla_\perp^2 n. \tag{7}$$

The momentum P in this case can be expressed through the density fluctuation n:

$$P=\int n^2 d\mathbf{r}>0, \tag{8}$$

and to leading order the energy ε coincides with P.

With the help of the inverse scattering transform the solution of the KP equation (7) was found explicitly in the form of a two-dimensional soliton: it is the so-called lump.[13] The momentum P on the lump is proportional to $\sqrt{1-v}$, so that $\partial P/\partial v<0$. For the NLSE (1) the existence of a soliton solution, similar to the lump was later confirmed numerically in Ref. 11. Also in that work, the whole family of two-dimensional solitons was found numerically. According to these results, the density well at the center of the soliton becomes deeper and deeper as the velocity decreases. There exists a velocity, v_{cr}, for which the density well reaches the "bottom," i.e., N becomes equal to zero. For smaller v this zero bifurcates, splitting into two separate zeros in the direction transverse to the direction of propagation of the soliton. These zeros correspond to two vortices with opposite circulations, resembling a vortex dipole. The reduction of the soliton velocity results in a growth of the distance ($l\simeq 1/v$) between the two vortices, so that in the small-velocity limit the dipole vortex pair is described to good accuracy by the Euler equation for incompressible fluids. The density fluctuations n for scales $\sim l$ are unimportant with respect to the phase variations. The density vanishes at the centers of each vortex and saturates sufficiently rapidly at distances of the core radius $a\sim 1$. Thus, the flow outside the core regions can to good accuracy be considered incompressible (see, e.g., Ref. 14):

$$\operatorname{div}\mathbf{U}=\nabla^2\phi=0. \tag{9}$$

The solution of this equation as $v\to 0$ can be written in the form

$$\phi(w)=\arg(w-il/2)+\arg(w+il/2),$$

where $w=x-vt+iy$. The main contribution to the energy in this limit is related to this incompressible flow,

$$\varepsilon \simeq 2\pi \log (1/v). \tag{10}$$

Using relation (6), one can write

$$\frac{\partial \varepsilon}{\partial v}\frac{\partial v}{\partial P}=v. \tag{11}$$

Introducing (10) we obtain

$$\frac{\partial P}{\partial v}=-\frac{2\pi}{v^2}<0. \tag{12}$$

Thus, in the limits of both small and large velocities the derivative $\partial P/\partial v$ is negative. If one assumes that the function $P(v)$ is monotonic, then it is readily seen that the derivative $\partial P/\partial v$ will be negative in the whole range of velocities v. Numerical integration of Eq. (2) confirms this assumption completely.[11] Thus, in one limit we have the KP solitons and, respectively, the KP equation, and in the other limit, for small velocities, we get two parallel vortex filaments with opposite circulations, which are similar to the vortex solutions of the 2D Euler equation.

3. The main purpose of the present paper is to investigate the stability of the whole family of two-dimensional soliton solutions. We assume that these solutions, representing stationary points of the Hamiltonian H for fixed momentum $\mathbf{P}$, should be stable in the 2D case, because both the KP and the Euler limits indicate their stability. In the first limit, the KP soliton realizes the minimum of the Hamiltonian for fixed $\mathbf{P}$ and therefore it is stable in accordance with the Lyapunov theorem.[2] For the Euler equation the fact that a two-point vortex distribution is stable is well known (see, for example, Ref. 15). We shall show that such solitons are unstable against three-dimensional perturbations.

Let us seek a solution of equation (1) in the form

$$\psi(\mathbf{r},t)=\psi_0(x',y)+\delta\psi\,(x'y,z,t), \tag{13}$$

where the soliton solution $\psi_0(x',y)$ obeys equation (2), $\delta\psi(x',y,z,t)$ is a small perturbation, and $x'=x-vt$. Let the perturbation depend on t and z in the following way:

$$\begin{pmatrix}\delta\psi\\ \delta\psi\,*\end{pmatrix}=\begin{pmatrix}u_1\\ u_2\end{pmatrix}\exp(-i\omega t+ikz).$$

Then, after linearization of equation (1) against the background of ψ_0, we arrive at the following spectral problem:

$$\omega\sigma_3 u-\tfrac{1}{2}k^2u+Lu=0. \tag{14}$$

Here

$$L=-iv\sigma_3\frac{\partial}{\partial x}+\frac{1}{2}(\partial_x^2+\partial_y^2)-\begin{pmatrix}2|\psi_0|^2-1 & \psi_0^2\\ \psi_0^{*2} & 2|\psi_0|^2-1\end{pmatrix}$$

is a Hermitian operator, and σ_3 is the Pauli matrix.

Since it is hardly possible to solve this spectral problem exactly, we shall restrict our analysis to the long-wavelength limit, where k is small compared with the inverse soliton size $1/l$, i.e., we introduce a small parameter $\varepsilon = kl \ll 1$. It means that the solution of the system (14) may be found in the form of a series in the small parameter ε:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \quad \omega = \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots . \tag{15}$$

To leading order we have

$$L u_0 = 0, \tag{16}$$

which shows that u_0 are neutral modes. Among them there are two modes corresponding to two independent infinitesimal translations of the soliton as a whole,

$$u_{01} = \frac{\partial}{\partial x} \begin{pmatrix} \psi_0 \\ \psi_0^* \end{pmatrix} \tag{17}$$

and

$$u_{02} = \frac{\partial}{\partial y} \begin{pmatrix} \psi_0 \\ \psi_0^* \end{pmatrix}. \tag{18}$$

Both modes are localized and belong to the bound states.These modes have different parities with respect to x and y. The function u_{01} is symmetric with respect to y, and u_{02} is antisymmetric. The neutral modes generate two independent branches with different parities, so that each branch can be considered separately.

The kernel of the operator L also contains an eigenfunction with zero eigenvalue; this is a neutral mode,

$$u_{03} = \begin{pmatrix} \psi_0 \\ -\psi_0^* \end{pmatrix},$$

corresponding to a small gauge transformation. This mode belongs to the continuous spectrum and therefore it is not interesting from the point of view of possible instability. It follows from first principles that unstable modes should be bounded. Modes which have a constant amplitude at infinity will evidently be stable. In the case of one-dimensional solitons there are only two eigenfunctions, connected with translation and gauge in the kernel of L. It is therefore natural to assume that in the 2D case the three functions given above will be present in the kernal of L.

In the next order of the perturbation expansion we obtain

$$\omega \sigma_3 u_0 + L u_1 = 0. \tag{19}$$

For symmetric perturbations this equation can easily be solved. Let us consider Eq. (2) for the stationary soliton and the complex conjugate of that equation. Differentiation of these equations with respect to v gives the equation

$$-i\sigma_3 u_{01} + L \frac{\partial}{\partial v} \begin{pmatrix} \psi_0 \\ \psi_0^* \end{pmatrix} = 0,$$

which coincides up to the constant factor $i\omega$ with equation (19) for $u_0 = u_{01}$. Hence, we have

$$u_{11}=i\omega\frac{\partial}{\partial v}\begin{pmatrix}\psi_0\\ \psi_0^*\end{pmatrix}. \tag{20}$$

The equation for the second order reads

$$\omega\sigma_3 u_1-\tfrac{1}{2}k^2u_0=-Lu_2. \tag{21}$$

The solvability condition for this equation is the orthogonality of its left-hand side to all functions from the kernel of L. For the given case, due to the parity, a nontrivial relation appears only for the function u_{01}. As a result, we have

$$\omega\langle u_{01}|\sigma_3|u_{11}\rangle=\tfrac{1}{2}k^2\langle u_{01}|u_{01}\rangle. \tag{22}$$

Inserting Eq. (20) into this expression and integrating by parts, we arrive at the dispersion relation

$$\omega^2=\frac{\varepsilon}{\partial P/\partial v}k^2<0. \tag{23}$$

We recall that $\partial P/\partial v<0$, as was shown in Sec. 2. Thus, the perturbation in question turns out to be unstable, with a growth rate Im ω given by Eq. (23). In the limit $v\to C_s$ this growth rate goes over to that for the instability of two-dimensional acoustic solitons in media with positive dispersion[2]:

$$\omega^2=\frac{P}{\partial P/\partial v}k^2=-2(1-v)k^2. \tag{24}$$

For the case of small $v\ll C_s$ the growth rate (23) is also simplified by means of (10) and (12),

$$\omega^2=-(kv)^2\log(1/v). \tag{25}$$

The instability governed by Eq. (24) represents an extension of the KP instability of 1D acoustic solitons, while instability (25) corresponds to the Crow instability for two parallel vortex filaments in ideal fluids.[6] In spite of the difference between these two physical situations, the causes of both instabilities are the same. As was stated in Sec. 1, if the velocity of a soliton decreases as its amplitude grows, one should expect instability with respect to transverse perturbations. It is important to note that this instability is of the self-focusing type, and it is expected that the instability saturates, if at all, at a level much higher than the initial amplitude. In the acoustic region the instability in the nonlinear stage initiates the collapse of acoustic waves.[16,17] For vortices this instability represents the first stage of the fundamental reconstruction of the flow topology, i.e., of the vortex reconnection (see recent numerical results[18]). It is also interesting to note that the general expression for the growth rate (23) does ont contain the logarithmic dependence on k, as follows from the results of Crow[6] for filaments of zero width.

Let us find the dispersion relation for antisymmetric perturbations. To find ω to leading order we must solve Eq. (19) but with u_0 replaced by u_{02} from Eq. (18). For this case the solution can also be found. Note that if one considers a soliton propagating at a small angle to the x axis, then the following relation may be derived:

$$-i\sigma_3 u_{02}+L\frac{\partial}{\partial v_y}\begin{pmatrix}\psi_0\\ \psi_0^*\end{pmatrix}\Big|_{v_y=0}=0. \qquad (26)$$

The derivatives with respect to v_y are easily expressed through the generator of the infinitesimal rotation,

$$\frac{\partial\psi_0}{\partial v_y}\Big|_{v_y=0}=-\frac{1}{v}[\mathbf{r}\times\nabla]\psi_0. \qquad (27)$$

As a result, the solution has the form

$$u_{12}=-\frac{i\omega}{v}[\mathbf{r}\times\nabla]\begin{pmatrix}\psi_0\\ \psi_0^*\end{pmatrix}. \qquad (28)$$

Next we replace u_{01} by u_{02} from (18) and u_{11} by u_{12} from (28) in Eq. (22) and integrate by parts. After simple algebra we obtain the following dispersion relation for an antisymmetric perturbation,

$$\omega^2=(kv)^2\frac{\int|\psi_y|^2 d\mathbf{r}}{Pv}=(kv)^2\frac{\varepsilon-Pv}{Pv}>0. \qquad (29)$$

Thus, the antisymmetric long-wavelength perturbations are stable over the whole range of soliton velocities, including both limits, i.e., for vortex filaments and for the 2D KP solitons. It should be noted that the frequencies (23) and (29) for the two limits go over to those obtained in Refs. 6 and 2.

4. The instability that we have found turns out to be of the self-focusing type, analogous to the instability of 1D (grey) solitons against transverse perturbations.[3] In the nonlinear stage a self-focusing tendency would lead to a division of 2D solitons or dipole vortices into separate cavities. For vortex filaments these cavities look like vortex rings. Such an assumption means that the process of cavity formation in this limit should be accompanied by the reconnection of vortex filaments. If initially the soliton distribution has no zeros this instability can be assumed to lead to the cavitation, i.e., to the appearance of a zero in the density profile, and, probably, at the later stages, to the birth of vortex rings. Recently the reconnection of vortex lines has been investigated numerically for Eq. (1) in three dimensions.[18] The main result was that vortex filaments of opposite "circulation" would reconnect whenever they come within a distance of a few core radii of each other. Further support for such scenario of instability development is the collapse of acoustic waves, which can be regarded as the nonlinear stage of the KP instability of solitons. The acoustic collapse, studied in detail both theoretically and numerically,[16,17] demonstrates the tendency toward catastrophic decrease in the density profile for solitons of small amplitude. Besides, recent experimental observations and a numerical study of the nonlinear development of the dark soliton instability revealed the formation of a point vortex stress,[19,20,22] similar to the von Karman street in fluids. Thus, all these facts support our hypothesis. In our opinion, it can be confirmed and proved by performing a three-dimensional numerical simulation.

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[1)]It is possible to show (see Ref. 11) that the usual expression for the momentum $P=\frac{i}{2}\int(\psi^*\nabla\psi-\psi^*\nabla\psi)d\mathbf{r}$ diverges logarithmically on the 2D soliton at infinity. The simplest renormalization leads to expression (5).

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